Quantum Systems’ Measurement through Product Hamiltonians

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0. Abstract

0.1. **Quantum systems’ measurement** is defined as the influence of the measured subsystem onto the measuring part, with the use of partial tracing, only.

0.2. The **results of a quantum measurement** are defined as classes of the following equivalence relation:

- Two states of the measured subsystem (say, “2”) are **equivalent**, if they imply the same reduced state of the measuring subsystem (say, “1”).
0.3. – the case of NON-SELECTIVE measurement is presented for the hamiltonians of the product form (possibly with more terms)

\[ H_{1,2} = H_1 \otimes I_2 + M_1 \otimes M_2 + N_1 \otimes N_2 + I_1 \otimes H_2. \]

- If it consists of multipliers (mutually commuting operators), then the maximal available in “1” information about the state of “2” is its diagonal.

- For noncommuting factors, the possible results of measurement are shown more informative.
0.4. – the case of \textsc{Intermediate} measurements is presented by examples of three-partite systems, with the interaction Hamiltonian of the form

\[ H_{1,2,3}^{\text{int}} = M_1 \otimes M_2 \otimes I_3 + I_1 \otimes N_2 \otimes N_3. \]

- If it consists of multipliers, then the results of the measurement of the state of part “3” by part “1” consists of one class (of all states of “3”).
- For non-commuting \( M_2 \) and \( N_2 \), some dependence happens.
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KEYWORDS: Schrödinger equation, multipartite systems, quantum measurement, product hamiltonian, reduced states
1. Basics - the quantum measurement

The presented notion of a QUANTUM MEASUREMENT requires the system to be divisible into at least two subsystems. Then the RESULT OF THE MEASUREMENT is assumed to be

- read from the state of the MEASURING PART
- caused by interactions with the MEASURED PART.
- calculated by reduction of the global state WITH THE USE OF PARTIAL TRACE with respect to the measured subsystem(s).

In this fashion, the measuring part plays the rôle of the OBSERVER.
1. Basics - quantum subsystems

Any $k$-parite system lives on the tensor product

\[(1.1) \quad \mathcal{H} = \bigotimes_{j=1}^{k} \mathcal{H}_j := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k\]

For a subset $A \subseteq \{1, 2, \ldots, k\}$, the tensor product

\[(1.2) \quad \mathcal{H}_A = \bigotimes_{j \in A} \mathcal{H}_j, \quad \langle \cdot, \cdot \rangle_A = \bigotimes_{j \in A} \langle \cdot, \cdot \rangle_j\]

is the Hilbert space of the compound subsystem
1. Basics - quantum states

A **quantum state** (simply: **state**) of a quantum system living on a Hilbert space $\mathcal{H}$ is any positive hermitian operator of trace-class defined on $\mathcal{H}$ with trace 1.

For a bi-partite system living on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the **reduced state of subsystem “1”** (…with respect to subsystem “2”) is defined as follows:

$$\rho|_1 = \text{Tr}_2 \rho,$$

where $\text{Tr}_2$ is the partial trace with respect to $\mathcal{H}_2$. 
A single \textbf{tool} for measurement is any unitary operator. We are interested in the actions of tools performed on a given state, which is then called the \textit{starting state}.

\textbf{Observed state of subsystem} “1” under tool $U$ is given by

\begin{equation}
\rho^{(U)} \big|_1 := \text{Tr}_2 \rho^{(U)} ,
\end{equation}

where

\[
\rho^{(U)} := U \rho U^\dagger .
\]
2. Definitions - the measured state

In the case of starting state $\rho$ of the PRODUCT FORM

\begin{equation}
\rho = \rho_1 \otimes \rho_2,
\end{equation}

the result of the reduction is called the OBSERVED STATE of "1" under tool $U$ AT THE MEASURED STATE $\rho_2$ of "2" and is denoted as follows

\begin{equation}
\rho_1^{(U)}[\rho_2] := \rho^{(U)}|_1 = \text{Tr}_2 \left( U \rho_1 \otimes \rho_2 \ U^\dagger \right)
\end{equation}
2. Definitions - the tool box

A TOOL-BOX for measurements of a system living on \( \mathcal{H} \) is any family \( \mathcal{U} := \{ U(t) : t \in T \} \) of unitary operators on \( \mathcal{H} \), indexed by elements of an arbitrary set \( T \).

Then the OBSERVED STATES are denoted simply by

\[
(2.4) \quad \rho_1^{(t)}[\rho_2] := \rho_1^{(U(t))}[\rho_2], \quad t \in T.
\]
2. Definitions - results of quantum measurements

- Two states $\rho_{2,I}$ and $\rho_{2,II}$ of subsystem “2” measured at state $\rho_1$ of subsystem “1” by the tool-box $U = \{U(t) : t \in T\}$ are said to be EQUIVALENT if the observed states are equal, i.e. satisfy the equality:

$$\rho_1^{(t)}[\rho_{2,I}] = \rho_1^{(t)}[\rho_{2,II}], \quad \text{for every } t \in T.$$

- The result of the measurement of the state of subsystem “2” at subsystem state $\rho_1$ by the tool-box $U$ is any class of equivalent states.
3. Bi-partite systems - chosen tool-boxes

Now, the tool-box is given by unitary operators

\begin{equation}
U(t) = \exp \left( -i t \hbar^{-1} H_{1,2} \right), \quad t \in T,
\end{equation}

with \( T \subset \mathbb{R} \), generated by a common hamiltonian

\begin{align}
H_{1,2} &= H_1 \otimes I_2 + H_{1,2}^{\text{int}} + I_1 \otimes H_2 \\
H_{1,2}^{\text{int}} &= M_1 \otimes M_2 + N_1 \otimes N_2 + \ldots,
\end{align}

finite sum
3. Bi-partite systems - product hamiltonians

REMARKS.

- Elements of $T$ are the INSTANTS OF OBSERVATIONS.

- Operators $H_1$ and $H_2$ describe the SELF-EVOLUTION of parts “1” and “2”, respectively.

- Operator $H_{1,2}^{\text{int}}$ describes a MULTI-TERM PRODUCT INTERACTION between the subsystems.
3. Bi-partite systems - chosen Hilbert spaces

The Hilbert spaces are chosen as follows

\[(3.3) \quad \mathcal{H}_1 = l^2_S \quad \text{and} \quad \mathcal{H}_2 = L^2(\mathbb{R}), \quad \text{card}(S) \leq \aleph_0 \]

where \( S \) is an at most countable set, i.e. the compound system Hilbert space equals

\[(3.4) \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = l^2_S \otimes L^2(\mathbb{R}) = L^2(S \times \mathbb{R}) \ldots \]

Then the reduced states of “1” and “2” are given by

\[(3.5) \quad \rho_{1, j, k}^{(t)} = \int_{\mathbb{R}} \rho_{j, k}^{(t)}(x, x) \, dx, \quad (j, k) \in S^2, \]

\[(3.6) \quad \rho_{2}^{(t)}(x, x') = \sum_{j \in S} \rho_{j, j}^{(t)}(x, x'), \quad (x, x') \in \mathbb{R}^2. \]
Abstract

Basic notions and main assumptions
The main definitions

Bi-partite systems

Intermediate measurements in three-partite systems
5. Closing remarks

1. Abstract

2. Basic notions

3. The main definitions

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   - Systems with commuting factors
   - Systems with non-commuting factors

5. Intermediate measurements in three-partite systems
   - Systems with commuting factors
   - Systems with non-commuting factors

6. 5. Closing remarks
3. Bi-partite systems - commuting factors

Assume, that the unitary operators \((U^{(t)}, t \in \mathbb{R})\) are multiplications by

\[
(3.7) \quad u_j^{(t)}(x) := \exp\{-i t [\omega_j + q_j \mu(x) + r_j \nu(x)] + \varepsilon(x)\},
\]

where \(q_j, r_j\) and \(\omega_j\) are real for \(j \in S\), and \(\mu, \nu\) and \(\varepsilon\) denote real Borel-measurable locally bounded functions on \(\mathbb{R}\). Hence, the hamiltonian is the multiplication by

\[
(3.8) \quad h_j(x) := \omega_j + q_j \mu(x) + r_j \nu(x) + \varepsilon(x).
\]
3. Bi-partite systems - commuting factors

Thus, entries of $\rho$, under action of the single tool at instant $t \in \mathbb{R}$ are equal to

$$
\rho_{j,k}^{(t)}(x, x') = \rho_{j,k}^{(0)}(x, x') \cdot \exp(i t W_{j,k}(x, x')) ,
$$

where for all $(j, k) \in S^2$ and almost everywhere in $(x, x') \in \mathbb{R}^2$, we have

$$
W_{j,k}(x, x') = (\omega_k - \omega_j) + (q_k \mu(x') - q_j \mu(x)) + (r_k \nu(x') - r_j \nu(x)) + (\varepsilon(x') - \varepsilon(x)) .
$$
3.1. Bi-partite systems - commuting factors

state of “1”

\[
\rho_1^{(t)}[\rho_2] = (\rho_1; j, k \ c_{2; j, k}^{(t)} : (j, k) \in S^2 ),
\]

(3.10) \[ c_{2; j, k}^{(t)} = \exp[i \ t (\omega_k - \omega_j)] \chi_{2; j, k}(t), \] where

\[
\chi_{2; j, k}(t) := \int_\mathbb{R} \exp\{i \ t [(q_k - q_j) \mu(x) + (r_k - r_j) \nu(x)]\} \rho_2(x, x) \, dx
\]

is the characteristic function of the following observable of subsystem “2”, treated as a random variable on \( \mathbb{R} \) equipped with the probability distribution determined by \( \text{diag}(\rho_2) \)

\[
Q_{2; j, k}(x) := (q_k - q_j) \mu(x) + (r_k - r_j) \nu(x), \quad x \in \mathbb{R},
\]
3.1. Bi-partite systems - commuting factors

state of “2”

\[
\rho_2^{(t)}[\rho_1] = (c_1^{(t)}(x, x') \rho_2(x, x') : (x, x') \in \mathbb{R}^2),
\]

(3.11) \(c_1^{(t)}(x, x') = \exp\left[i t (\varepsilon(x') - \varepsilon(x))\right] \chi_1(t; x, x'),\) where

\[
\chi_1(t; x, x') := \sum_{j \in S} \exp\{it [q_j (\mu(x') - \mu(x)) + r_j (\nu(x') - \nu(x))]\} \rho_{1;j,j},
\]

is the characteristic function of the multiplier on \(\mathcal{H}_1\)

\[
Q_{1;j}(x, x') = [q_j (\mu(x') - \mu(x)) + r_j (\nu(x') - \nu(x))].
\]

It is a random variable on \(S\), where atom \(j\) has probability \(\rho_{1;j,j}\).
3.1. Bi-partite systems - commuting factors

**Main corollary**

The Main Corollary.

The information about the measured state $\rho_2$ [state $\rho_1$] available for subsystem “1” [subsystem “2”] through a tool-box given by (3.7) cannot exceed the distribution determined by the diagonal $\text{diag}(\rho_2)$ [or, by the diagonal $\text{diag}(\rho_1)$, respectively].

This remains true for any hamiltonian of the product form (3.2), independently of the number of terms, as long as all terms with the same index commute. The details are omitted.
3.2. Bi-partite systems - non-commuting factors

Now we assume, that the hamiltonian is of the form,

\[ \hbar^{-1}(H_{1,2}\Psi)_j(x) := (\omega_j + q_j x)\psi_j(x) + iv_j\psi_j'(x), \quad j \in S, \quad x \in \mathbb{R}. \]

for \( \Psi = (\psi_j : j \in S) \), with square-integrable AND differentiable components \( \psi_j \in H_2 = L^2(\mathbb{R}) \). Then, by [3, Lemma 1], for ARBITRARY initial pure state represented by

\[ \Psi^{(0)} = (\psi_j^{(0)}(x) : j \in S, x \in \mathbb{R}) \in \mathcal{H}, \]

\[ \psi_j^{(t)}(x) = \psi_j^{(0)}(x + tv_j) \exp \left[ -i\omega_j t - iq_j tx - \frac{i}{2}q_j v_j t^2 \right]. \]
3.2. Bi-partite systems - non-commuting factors

Accordingly, the density matrix of the system varies as follows:

\[
\rho_{j,k}^{(t)}(x, x') = \exp \left\{ i \left[ (\omega_k - \omega_j) + (q_k x' - q_j x) \right] t + \right.
\]
\[
\left. + \frac{i}{2} (q_k v_k - q_j v_j) t^2 \right\} \rho_{j,k}^{(0)}(x + tv_j, x' + tv_k)
\]

and the reduced states of subsystems “1” and “2” depend on time \( t \) as follows.
3.2. Bi-partite systems - non-commuting factors

state of “1”

(3.13)

\[
\rho_{1;j,k}^{(t)} = \exp \left[ i (\omega_k - \omega_j) t + \frac{i}{2} (q_k v_k - q_j v_j) t^2 \right] \rho_{1;j,k}^{(0)}(x + tv_j, x + tv_k) dx.
\]

In particular, under the product form (2.2), we have

(3.14)

\[
\rho_{1;j,k}^{(t)} = \exp \left[ i (\omega_k - \omega_j) t + \frac{i}{2} (q_k v_k - q_j v_j) t^2 \right] \rho_{1;j,k}^{(0)}
\times \int_{\mathbb{R}} \exp \left[ i (q_k - q_j) x t \right] \rho_{2}^{(0)}(x + tv_j, x + tv_k) dx.
\]
3.2. Bi-partite systems - non-commuting factors

state of “1”

The information about the starting state $\rho_2^{(0)} = \rho_2$ within the reduced states $\rho_1^{(t)}$ is encoded in the coefficients

$$\Phi_{j,k}^{(t)}[\rho_2] := \int_{\mathbb{R}} \exp \left[i(q_k - q_j) x \right] \rho_2(x + tv_j, x + tv_k) \, dx.$$ 

Thus, for sufficiently large set of values of the differences $q_k - q_j$ and $v_k - v_j$, $j, k \in S$, the kernel $\rho_2$ can be completely restored, whenever it is continuous. This is a consequence of noncommutativity of multiplication by $x$ and the momentum operator $i\hbar \frac{d}{dx}$ in the hamiltonian.
3.2. Bi-partite systems - non-commuting factors

State of “2”

\[ \rho_2^{(t)}(x, x') = \sum_{k \in S} \exp \left[ i q_k (x' - x) t \right] \rho_0^{(0)}(x + tv_k, x' + tv_k) . \]

In particular, under the product form (2.2), we have

\[ (3.15) \quad \rho_2^{(t)}(x, x') = \sum_{k \in S} \exp \left[ i q_k (x' - x) t \right] \times \]
\[ \times \sum_{k \in S} \exp \left[ i q_k (x' - x) t \right] \rho_1^{(0)}(x + tv_k, x' + tv_k) . \]
Thus, starting states $\rho_{1, \Pi}^{(0)}$ and $\rho_{1, \Pi}^{(0)}$ of “1”, measured at state $\rho_{2}^{(0)}$ of “2” with $(U^{(t)}; t \in T)$, are equivalent, if for all $t \in T$, $x, x' \in \mathbb{R}$ we have

$$\sum_{k \in S} \rho_{1, \Pi; k, k} \chi_{k}^{(t)}(x, x') = \sum_{k \in S} \rho_{1, \Pi; k, k} \chi_{k}^{(t)}(x, x')$$

$$\chi_{k}^{(t)}(x, x') := \exp \left[ i q_k (x' - x) t \right] \rho_{2}^{(0)}(x + t v_k, x' + t v_k)$$

Thus, the maximal information available for “2” is the probability distribution given by the diagonal $\text{diag} \left( \rho_{1}^{(0)} \right)$. This is due to commutativity of multiplications by $\omega, q$ and $v$ in the hamiltonian.
4. Three-partite systems with product interaction

In this section we assume equality (1.1) with $k = 3$, where the factor-spaces are chosen as follows

\[(4.1) \quad \mathcal{H}_1 = l^2_S \quad \text{and} \quad \mathcal{H}_2 = \mathcal{H}_3 = L^2(\mathbb{R}),\]

and $S$ is again an at most countable set. Thus,

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 = l^2_S \otimes L^2(\mathbb{R}^2) = L^2(S \times \mathbb{R}^2).
\]
4. Three-partite systems with product interaction

For the tool-box of a measurement we assume

\[ U^{(t)} = \exp \left(-i t \hbar^{-1} H_{1,2,3} \right), \quad t \in T, \]

generated by a common hamiltonian

\[ H_{1,2,3} = H_1 \otimes I_2 \otimes I_3 + I_1 \otimes H_2 \otimes I_3 + I_1 \otimes I_2 \otimes H_3 + H_{1,2,3}^{\text{int}} \]

with\(^1\)

\[ H_{1,2,3}^{\text{int}} = H_{1,2} \otimes I_3 + I_1 \otimes H_{2,3}. \]

\(^1\) We apply the convention, that \( K_A \) stands for an operator defined on \( \mathcal{H}_A \), for any subset \( A \subset \{1, 2, 3\} \).
4. Three-partite systems with product interaction

In our assumptions, subsystems “3” and “1” have no immediate interaction. Therefore we interpret the implied dynamics $U(t) = \exp(-itH_{1,2,3})$ as a tool-box for \textbf{INTERMEDIATE MEASUREMENTS} between subysystems “1” and “3” (\textit{via} subsystem “2”). For simplicity we confine the considerations to

\begin{equation}
H_{1,2,3}^{\text{int}} = M_1 \otimes M_2 \otimes I_3 + I_1 \otimes N_2 \otimes N_3.
\end{equation}

\footnote{The below given subclasses of interactions are suitably modified hamiltonians of Section 3. Accordingly, the solutions obtain a simply verifiable form.}
4. Three-partite systems with product interaction

REMARK. Now the reduced states of subsystems “1”, “2” and “3” are given by the following density matrices

\[
\rho_{1;j,k}^{(t)} = \int_{\mathbb{R}^2} \rho_{j,k}^{(t)}(x, y; x, y) \, dx \, dy, \quad j, k \in S,
\]

\[
\rho_{2}^{(t)}(x, x') = \int_{\mathbb{R}} \sum_{j} \rho_{j,j}^{(t)}(x, y; x', y) \, dy, \quad x, x' \in \mathbb{R},
\]

\[
\rho_{3}^{(t)}(y, y') = \int_{\mathbb{R}} \sum_{j} \rho_{j,j}^{(t)}(x, y; x, y') \, dx, \quad y, y' \in \mathbb{R}.
\]
4.1. Three-partite systems - commuting factors

For this subsection the particular subclass is defined by the assumption, that the tool-box is the group of unitary operators $U^{(t)}$ equal to the multiplication by the functions $u_j^{(t)}(x, y)$ equal to

\[(4.6)\]
\[
\exp(-i \, t \, (\omega_j + \varepsilon_2(x) + \varepsilon_3(y) + m_j \, \mu(x) + \nu_2(x) \nu_3(y))),
\]

for $j \in S$, $x, y \in \mathbb{R}$. 

4.1. Three-partite systems - commuting factors

Consequently, entries of $\rho^{(t)}$ under action of the single tool $U^{(t)}$ (or, at instant $t \in \mathbb{R}$) are equal to

$$\rho_{j,k}^{(t)}(x,y;x',y') = \rho_{j,k}^{(0)}(x,y;x',y') \cdot \exp(i t W_{j,k}(x,y;x',y')),$$

where $W_{j,k}(x,y;x',y')$ equals

$$(\omega_k - \omega_j) + (\varepsilon_2(x') - \varepsilon_2(x)) + (\varepsilon_3(y') - \varepsilon_3(y)) +
+ (m_k \mu(x') - m_j \mu(x)) + (\nu_2(x') \nu_3(y') - \nu_2(x) \nu_3(y)),$$

and this holds for all $(j,k) \in S^2$ and for almost all $(x,y;x',y')$ in $\mathbb{R}^4$. 
In the case of commuting factors, if the starting state $\rho$ of the system is of the product form

$$\rho_1 \otimes \rho_2 \otimes \rho_3,$$

where $\rho_j$ acts on $\mathcal{H}_j$, $j = 1, 2, 3$,

then the reduced evolution of subsystems “1”, “2” and “3” are given by the multiplication of the starting density matrices term-by-term by suitable matrices $c_{2,3}^{(t)}$, $c_{1,3}^{(t)}$ and $c_{1,2}^{(t)}$, respectively, as follows.
4.1. Three-partite systems - commuting factors

state of “1”

Observed state of system “1” equals

$$\rho_1^{(t)}[\rho_2 \otimes \rho_3] = \rho_1 \otimes c_{2,3}^{(t)} := (\rho_{1;j,k} c_{2,3;j,k}^{(t)} : (j, k) \in S^2),$$

where $c_{2,3;j,k}^{(t)} = \exp(i t (\omega_k - \omega_j)) \chi_{2,3;j,k}(t)$, with

(4.8) $\chi_{2,3;j,k}(t) := \int_{\mathbb{R}^2} \exp(i t [(m_k - m_j) \mu(x)])$

$$\times \rho_2(x, x) \rho_3(y, y) \, dx \, dy$$

$$= \int_{\mathbb{R}} \exp(i t [(m_k - m_j) \mu(x)]) \rho_2(x, x) \, dx.$$
4.1. Three-partite systems - commuting factors

The observed state of “2” at instant \( t \in \mathbb{R} \) equals

\[
\rho_2^{(t)}[\rho_1 \otimes \rho_3] = \rho_2 \odot c_{1,3}^{(t)} := \left( \rho_2(x, x') c_{1,3}^{(t)}(x, x') \right)
\]

where for \((x, x') \in \mathbb{R}^2, t \in \mathbb{R}\),

\[
c_{1,3}^{(t)}(x, x') = \exp(i t (\varepsilon_2(x') - \varepsilon_2(x))) \chi_1(t; x', x), \quad \text{with}
\]

\[
\chi_1(t; x, x') := \sum_{j \in S} \int_{\mathbb{R}} \exp\{i t [m_j (\mu(x') - \mu(x)) + (\nu_2(x') - \nu_2(x)) \nu_3(y)]\} \rho_{1;j,j} \rho_3(y, y) dy,
\]
4.1. Three-partite systems - commuting factors

state of “2”

\( \chi_{1,3}(.; x, x') \) is the characteristic function of the following multiplier on \( \mathcal{H}_1 \otimes \mathcal{H}_3 \)

\[
O_{1,3;j,y}(x, x') = [q_j (\mu(x') - \mu(x)) + (\nu_2(x') - \nu_2(x))\nu_3(y)], \quad j \in S, \ y \in \mathbb{R}
\]

treated as the random variable on \( S \times \mathbb{R} \), with the product measure, where atom \( j \) has probability, determined by \( \text{diag}(\rho_1) \) and the variable \( y \) possesses p.d.f. given by \( \text{diag}(\rho_3) \) as follows,

\[
(4.9) \quad F_3(a) = \int_{(-\infty,a]} \rho_3(y, y) \, dy, \quad a \in \mathbb{R}.
\]
4.1. Three-partite systems - commuting factors

state of “3”

Finally, for the reduced state of subsystem “3” we obtain

\[
\rho_3^{(t)}[\rho_1 \otimes \rho_2] = \rho_3 \circ c_{1,2}^{(t)} := \left( \rho_3(y, y') c_{1,2}^{(t)}(y, y') \right)
\]

where for \((y, y') \in \mathbb{R}^2, t \in \mathbb{R},\)

\[
c_{1,2}^{(t)}(y, y') = \exp(i t (\varepsilon_3(y') - \varepsilon_3(y))) \chi_{1,2}(t; y, y'),
\]

and the latter function is given by

\[
\chi_{1,2}(t; y, y') = \int_{\mathbb{R}} \exp\{i t [\nu_2(x) (\nu_3(y') - \nu_3(y))]) \} \rho_2(x, x) dx.
\]
4.1. Three-partite systems - commuting factors

state of “3”

Thus, \( \chi_{1,2} \) is the characteristic function of another observable of subsystem “2”

\[
O_2(y, y'; x) := \nu_2(x)(\nu_3(y') - \nu_3(y)) \quad x \in \mathbb{R},
\]

treated as a random variable on \( \mathbb{R} \) equipped with the probability distribution function (p.d.f.) determined again by the diagonal \( \text{diag}(\rho_2) \) by

\[
F_2(a) = \int_{(-\infty,a]} \rho_2(x, x) \, dx
\]

OBSERVED STATES OF “3” ARE INDEPENDENT OF \( \rho_1 \).
4.2. Three-partite systems - non-commuting factors

In this section we modify the hamiltonian of subsection 3.2:
(4.10)
\[ \hbar^{-1} (H_{1,2,3} \psi)_j (x, y) := (\omega_j + m_j x) \psi_j (x, y) + iy \frac{\partial}{\partial x} \psi_j (x, y), \]
whenever \( \Psi = (\psi_j : j \in S) \), with \( \psi_j \in L^2(\mathbb{R}^2) \), \( j \in S \), \( x \in \mathbb{R} \).

For consistency with (4.3) – (4.5), in particular with
\[ H_{1,2,3}^{\text{int}} = M_1 \otimes M_2 \otimes I_3 + I_1 \otimes N_2 \otimes N_3, \]
the operators can be chosen as follows, \( x \in \mathbb{R} \),
4.2. Three-partite systems - non-commuting factors

\[ H_1(\pi) = (\hbar \omega_j \pi_j : j \in I), \quad H_2\psi(x) = H_3\psi(x) = 0, \]
\[ M_1(\pi) = (m_j \pi_j : j \in S), \quad M_2\psi(x) = \hbar x \psi(x), \]
\[ N_2\psi(x) = i \hbar \psi'(x), \quad N_3\psi(y) = y \psi(y) \]

Proceeding as for Lemma 1 in [3, Section 3], one can prove that the time dependent solution \( \Psi(t) = (\psi_j(t) : j \in S) \) to the Schrödinger equation equals
4.2. Three-partite systems - non-commuting factors

\[ \psi_j^{(t)}(x, y) := 3 \]

\[ := \psi_j^{(0)}(x + ty, y) \exp \left[ -i\omega_j t - i m_j tx - \frac{i}{2} m_j y t^2 \right], \]

for arbitrary initial pure state represented by

\[ \Psi^{(0)} = \left( \psi_j^{(0)}(x, y) : j \in S, (x, y) \in \mathbb{R}^2 \right) \in \mathcal{H}. \]

Hence, the density matrix of the system varies as follows:

\[ ^3 \text{In order to avoid the derivation of the solution, it suffices to check that } i \frac{\partial}{\partial t} \psi_j^{(t)}(x) = (H_{1,2,3} \Psi^{(t)})_j(x, y), \text{ for differentiable } \Psi^{(0)}. \]
4.2. Three-partite systems - non-commuting factors

\[
\rho_{j,k}^{(t)}(x, y; x', y') = \\
= \exp \{ i \left[ (\omega_k - \omega_j) + (m_k x' - m_j x) \right] t \} \times \\
\times \exp \left\{ \frac{i}{2} (m_k y' - m_j y) t^2 \right\} \rho_{j,k}^{(0)}(x + ty, y; x' + ty', y'),
\]

and the reduced states of subsystems “1” and “3” depend on time \( t \) as follows.
4.2. Three-partite systems - non-commuting factors: state of “1”

\[
\rho_1^{(t)} [\rho_2 \otimes \rho_3] = \rho_{1:j,k}^{(0)} \exp \{ i \ t (\omega_k - \omega_j) \} \\
\times \int_{\mathbb{R}^2} \exp \left\{ i \ t (m_k - m_j) (x + \frac{1}{2}yt) \right\} \\
\times \rho_{2}^{(0)} (x + ty, x + ty) \rho_{3}^{(0)} (y, y) \ dx \ dy
\]

\[
= \rho_{1:j,k}^{(0)} \exp \{ i \ t (\omega_k - \omega_j) \} \\
\times \int_{\mathbb{R}^2} \exp \left\{ i \ t (m_k - m_j) (x - \frac{1}{2}yt) \right\} \\
\times \rho_{2}^{(0)} (x, x) \rho_{3}^{(0)} (y, y) \ dx \ dy
\]
4.2. Three-partite systems - non-commuting factors: state of “1”

\[ (4.12) \quad \rho_1^{(t)} = (\rho_{1;j,k} c_{2,3;j,k}^{(t)} : (j, k) \in S^2) \]

where for \((j, k) \in S^2, t \in \mathbb{R},\)

\[ (4.13) \quad c_{2,3;j,k}^{(t)} = \exp[i \, t \, (\omega_k - \omega_j)] \chi_{2,3;j,k}(t), \]

with \(\chi_{2,3;j,k}(t)\) given by

\[ \int_{\mathbb{R}^2} \exp\{i \, t \, [(m_k - m_j)(x - \frac{1}{2}yt)]\} \rho_2^{(0)}(x, x) \rho_3^{(0)}(y, y) \, dx \, dy \]
4.2. Three-partite systems - non-commuting factors: state of “1”

Now, \( \chi_{2,3; j, k}(t) \) is the value of the characteristic function of the observable of the compound subsystem “2, 3”:

\[
O_{2;j,k}(x, y) := (m_k - m_j) \left( x - \frac{1}{2} y t \right), \quad x, y \in \mathbb{R},
\]

treated as a random variable on \( \mathbb{R}^2 \) equipped with the product of the probability distributions determined by the diagonals \( \text{diag}(\rho_2^{(0)}) \) and \( \text{diag}(\rho_3^{(0)}) \) as given above.

The observed state of “1” depends on \( \rho_3 \) through \( \text{diag}(\rho_3) \), at most.
4.2. Three-partite systems - non-commuting factors: state of “2”

One easily gets, that for \((x, x') \in \mathbb{R}^2, t \in \mathbb{R},\)

\[
\rho_2^{(t)}(x, x') = \sum_{j \in S} \exp\{i t m_j (x' - x)\} \rho_{1; j, j} \\
\times \int_{\mathbb{R}} \rho_2(x - ty, x' - ty) \rho_3(y, y) \, dy,
\]

(4.14)

**OBSERVED STATE OF “2” DEPENDS ON \(\rho_1\) AND \(\rho_3\)**
4.2. Three-partite systems - non-commuting factors: state of “3”

Finally, for the reduced state of subsystem “3” we obtain

\[(4.15) \quad \rho_3^{(t)}[\rho_1 \otimes \rho_2] = \left( \rho_3^{(0)}(y, y') c_{1,2}^{(t)}(y, y') \right)\]

where for \((y, y') \in \mathbb{R}^2, t \in \mathbb{R}\), the coefficients equal

\[
c_{1,2}^{(t)}(y, y') = \sum_{j \in S} \int_{\mathbb{R}} \exp \left[ \frac{i}{2} (m_j(y' - y) t^2 \right] \times \rho_1^{(0)}(x + ty, x + ty') dx .
\]
4.2. Three-partite systems - non-commuting factors: state of "3"

or, equivalently

$$\rho_3^{(t)}(y, y') = \rho_3^{(0)}(y, y') \sum_{j \in S} \exp \left[ \frac{i}{2} (m_j(y' - y) t^2 \right] \rho_1^{(0)}_{1; j, j}$$

$$\times \int_{\mathbb{R}} \rho_2^{(0)}(x, x + t(y' - y)) \, dx.$$

**The observed state of “3” depends on $\rho_1$ through $\text{diag}(\rho_1)$, at most.**
5. Closing remarks

In rough description, in our model the tools are observables built into the hamiltonian and we allow them to act all the time during the evolution. In this fashion we have modelled the influence of the state of the measured system upon the changes of the measuring subsystem state. This makes our considerations close to the fashion presented e.g. by Žurek [12] and Attal [2]). Our main goal was to point at the influence of commutativity of the factors in the interaction.
5. Closing remarks

It is worth to stress, that in the traditional system of notions, the model of measurement is based on the average values of observables, which goes back to the formulation of quantum mechanics by von Neumann [10]. In such cases, the problem of minimal sets of the observables is solved in the works by Jamiołkowski [5], [6] and other authors (see the survey paper [8] and the references therein).

For the presented model, the set of tools (including instants of observations) sufficient for identification of the measured state becomes the main problem for investigations. Also the question of the subject of measurements is open. For instance, can we use starting states which are *not* of the product form.
5. Closing remarks

The relation to the entropy analysis, maximal entanglement and other similar problems, considered for instance in [11] is omitted. Also questions extended over compound systems of large number of parts (as in [15]) is not even touched here, since the basic features do not require more than 3 parts.

On the other hand, the proposed definition of measurement seems to be use-less in cases, when the system lives on a product of Hilbert spaces, but the **set of all possible states** does not cover the space of all trace class positive operators with trace 1.
5. Closing remarks - bad news

EXAMPLE. Let the Hilbert space of a bi-partite system \( \mathcal{H}_{1,2} \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \) be spanned by 4-dimensional unit vectors

\[
\psi(\sigma_1, \sigma_2) = U(\sigma_1, \sigma_2) \cdot a_{\sigma_1}, \quad (\sigma_1, \sigma_2) \in \{-, +\}^2, \quad U(\sigma_1, \sigma_2), a_{\sigma_1} \in \mathbb{C},
\]

with

\[
U(\sigma_1, \sigma_2) = \frac{1 - \sigma_1 \cdot \sigma_2 \exp(i\alpha)}{2}, \quad \sigma_1, \sigma_2 \in \{-, +\} \quad \alpha \in \mathbb{R}.
\]

\( \mathcal{H}_{1,2} \) is of complex dimension 2. The corresponding reduced density matrix is diagonal, and independently of \( \alpha \) we have

\[
\rho|_1(+) = \text{tr}_2 \rho(+, +) = |a_+|^2, \quad \rho|_1(−) = |a_−|^2.
\]


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