LARGE DEVIATIONS

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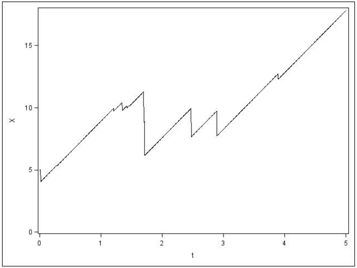
Historical motivation: Ruin problem

Lundberg (1903). Cramér-Lundberg process:

$$X_t = u + ct - \sum_{i=1}^{N_t} \zeta_i$$

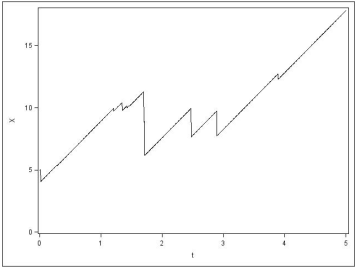
X_t =capital of insurance company at time t,
c =premiums income,
$\{\zeta_i\}$ claims losses.



Ruin problem (cont.)

Let $\Psi(u) = \mathbf{P}\left\{X_t < 0, \text{ for some } t \geq 0 \middle| X_0 = u\right\}$.
Lundberg (1903), Cramér (1930):

$$\Psi(u) \sim Ce^{-Ru} \quad \text{as } u \to \infty.$$



Ruin problem (cont.)

Why is this a large deviations problem?

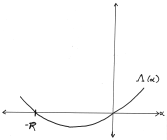
First, exponential decay as $u \to \infty$; in particular,

$$\lim_{u \to \infty} \frac{1}{u} \log \mathbf{P}\{\text{ruin}\} = -R,$$

where R is the *large deviation rate function*.

Second, proof is based on a change-of-measure argument.
Shifted measure: If $X_1 \sim \mu$, then consider

$$\mu_R(dx) := e^{-Rx} \mu(dx).$$



Statistics: sample means

$X_1, X_2, \ldots$ i.i.d. $\subset \mathbb{R}$.
Mean $\mathbf{E}[X_1] = m$.
Sample mean:

$$\bar{X}_n := \frac{X_1 + \cdots + X_n}{n}.$$

Law of large numbers:

$$\mathbf{P}\left\{|\bar{X}_n - m|\right\} \to 0 \quad \text{as} \quad n \to \infty.$$

How fast?
E.g., what is

$$\lim_{n\to\infty} \mathbf{P}\left\{\bar{X}_n > a\right\} \quad \text{for} \quad a > m?$$

Statistics: sample means

Central limit approximation:
Let $S_n = X_1 + \cdots + X_n$.
Then by the central limit theorem,

$$\bar{X}_n = \frac{S_n}{n} \approx \mathrm{Normal}\left(m, \frac{\sigma^2}{n}\right).$$

How good is this estimate?

Example

Suppose $\{X_i\}$ is i.i.d. Bernoulli sequence; thus,

$$X_i = \begin{cases} 1 & \text{w.p. } 1/2, \\ 0 & \text{w.p. } 1/2. \end{cases}$$

Then $S_n = X_1 + \cdots + X_n \sim$ binomial.
Now according to the central limit theorem, since $m = \sigma = 1/2$,

$$\begin{aligned} \mathbf{P}\left\{\frac{S_n}{n} > a\right\} &= \mathbf{P}\left\{\frac{S_n - n \cdot \frac{1}{2}}{\frac{1}{2}\sqrt{n}} > \sqrt{n}(2a-1)\right\} \\ &\approx 1 - \Phi\left(\sqrt{n}(2a-1)\right). \end{aligned}$$

But if $a > 1$, this probability is actually $= 0$.

Cramér's classical estimate

Assume finite moment generating fn.: $\lambda(\alpha) := \mathbf{E}\left[e^{\alpha X_i}\right] < \infty$.

Theorem (Cramér, 1938)

For any $a > m$,

$$\mathbf{P}\left\{\frac{S_n}{n} > a\right\} \sim \frac{c}{\sqrt{n}} e^{-nI(a)} \quad \text{as} \quad n \to \infty.$$

Shows:

- Exponential decay.
- Decay according to a "rate function" $I(x)$ (the convex conjugate of $\log \lambda$). This rate function will be *different* for every μ, where $X_i \sim \mu$.

Empirical measures

$\{X_i\}$ i.i.d., $X_i : \Omega \to \mathbb{S} = \{a_1, \ldots, a_d\}$.
Empirical measure:

$$L_n(a_j) = \frac{1}{j} \sum_{i=1}^{n} \delta_{X_i}(a_j),$$

that is, the proportion of observed samples which $= a_j$.
Note: L_n is a probability measure on $\mathbb{S}$.

Question: How likely will we empirically observe an $L_n(\cdot)$,
given the true distribution function.

Empirical measures (cont.)

Sanov (1957): If $\mathcal{S}$ is a set of empirical measures, then

$$\mathbf{P}\left\{L_n \in \mathcal{S}\right\} \approx e^{-nI(\mathcal{S})};$$

more precisely,

$$\lim_{n\to\infty} \log \mathbf{P}\left\{L_n \in \mathcal{S}\right\} = -I(\mathcal{S}).$$

Related problem: Brownian motion in a “tube" (Shilder, 1966). Consider

$$\mathbf{P}\left\{B_n(t) \in \Gamma(t) : 0 \leq t \leq 1\right\}, \quad n \to \infty,$$

where $B_n(t) = \frac{1}{\sqrt{n}} B(t)$ and $B(t)$ is Brownian motion.

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