The stochastic modeling 6

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A schedule of the lecture

- Hitting times
- Reflection principle
- Hitting time of a two sided barrier for BM
- Dubins Schwarz's Theorem
- Mixed Processes

Hitting times

Let W_t be a Brownian motion. For W_t we denote by T_a the first hitting time of a level (a barrier) $a > 0$ by W_t, i.e.

$$T_a = \inf\{t > 0 : W_t = a\}.$$

Program 39. T_a is a random variable such that for every scenario $\omega \in \Omega$,

$$T_a(\omega) = t,$$

where $W_t(\omega) = a$ is the first moment of reaching the level a.
Question: what is a distribution of RV T_a for given a? Let us create an empirical distribution of T_a for $a = 0.5$. We observed that there is a chance that a trajectory does not reach the level a. If we observe trajectories in time interval $[0, 10]$ and if a trajectory does not reach the level $a = 0.5$ then in this case we will put $t = 10.5$. Program 40.

Reflection principle

To calculate the distribution of T_a we need to use a reflection principle. Let us define the maximum process of the Brownian motion by

$$M_t = \sup_{s \le t} W_t.$$

Note that for $y \ge 0$

$$\{M_t \ge y\} = \{T_y \le t\}.$$

Obviously if $T_y(\omega) \le t$ it means that the Brownian motion reached the level y to the time t, hence the maximum $M_t \ge y$.

Theorem

For $y \ge 0$, $x \le y$ we have

$$P(W_t \le x, M_t \ge y) = P(W_t \ge 2y - x).$$

Consequently if we take in the above formula $x = y$ then for $y \ge 0$

$$P(W_t \le y, M_t \ge y) = P(W_t \ge y).$$

The distribution function of hitting time

On the other hand for $y \geq 0$

$$P(W_t > y, M_t \geq y) = P(W_t > y) = P(W_t \geq t).$$

The sets $\{W_t > y, M_t \geq y\}$ and $\{W_t \leq y, M_t \geq y\}$ are disjoint. Thus for $y \geq 0$

$$P(M_t \geq y) = P(W_t > y, M_t \geq y) + P(W_t \leq y, M_t \geq y) = 2P(W_t > y).$$

But $W_t \in N(0, \sqrt{t})$. Hence for $y \geq 0$

$$P(W_t > y) = P(W_t/\sqrt{t} > y/\sqrt{t}) = 1 - \Phi(y/\sqrt{t}),$$

where Φ is a distribution function of the standard normal distribution. Hence for $y \geq 0$

$$F_y(t) = P(T_y \leq t) = P(M_t \geq y) = 2(1 - \Phi(y/\sqrt{t}))$$

The density function of hitting time

The density function for $t \geq 0$

$$f(t) = F_y'(t) = \phi(y/\sqrt{t})yt^{-3/2},$$

where ϕ is the density function of the standard normal distribution. Recall

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

We can compare the empirical distribution with the theoretical one. Program 41 $y = 0.5$. For given $y > 0$ the expected time of reaching the barrier y is infinity i.e.

$$ET_y = \int_0^\infty \frac{t}{2\pi}e^{-y^2/(2t)}\frac{y}{t^{3/2}}dt \sim \int_1^\infty \frac{1}{\sqrt{t}}dt = \infty,$$

There is not a singularity at $t = 0$, since

$$\lim_{t\to 0^+} \frac{e^{-1/t}}{\sqrt{t}} = \lim_{u\to\infty} e^{-u}\sqrt{u} = 0.$$

Hitting time of a two sided barrier for BM

Let $a < 0 < b$ and T_a, T_b the two hitting times. Recall

$$T_a = \inf\{t > 0 : W_t = a\}.$$

Use Program 42 and 43 to see the following phenomena

Theorem

Let $T^* = \min\{T_a, T_b\}$ *then*

$$P(T_a < T_b) = \frac{b}{a-b}.$$

Moreover

$$ET^* = -ab.$$

Compare two cases $|a| > |b|$ and $|a| < |b|$. The density function is a rather complicated.

Hitting time- Ornstein -Uhlenbeck processes

Recall that $X_t = -\theta X_t dt + dW_t$, $X_0 = x > 0$ is OU processes, where $\mu = 0$, $\sigma = 1$. Let

$$T_0^{OU} = \inf\{t \geq 0 : X_t = 0\}.$$

Theorem

For any $x > 0$ the density function of T_0 equals

$$f(t) = A'(t)\phi(-2x/\sqrt{A(t)})\frac{x}{A(t)^{3/2}},$$

where

$$A(t) = \int_0^t e^{2ks} ds$$

and ϕ is the density function of $N(0, 1)$.

Proof

The solution OU is given by

$$X_t = e^{-\theta t}(x + \int_0^t e^{ks} dW_s).$$

Thus

$$T_0^{OU} = \inf\{t \geq 0 : x + \int_0^t e^{ks} dW_s = 0\}.$$

It is known that for a function f such that

$$\int_0^t f^2(s) ds < \infty, \quad \text{for all} \quad t \geq 0$$

the processes

$$M_t = \int_0^t f(s) dW_s$$

is a martingale.

Martingale =Dubins Schwarz's Theorem

A continuous martingale M such that

$$< M >_\infty = \int_0^\infty f^2(s)ds = \infty$$

is a time changed of Brownian motion. In other words, there exists a Brownian motion W such that

$$M_t = W_{<M>_t} = W_{B(t)},$$

where

$$B(t) = \int_0^t f^2(s)ds.$$

From Dubins Schwarz's Theorem there is BM W such that

$$T_0^{OU} = \inf\{t \geq 0 : W_{A(t)} = -x\}.$$

Consequently

$$P(A(T_0^{OU}) \leq t) = P(T_0^{OU} \leq A^{-1}(t)) = P(T_{-x} \leq t).$$

Mixed Processes involving Compound Poisson Processes

Let X be a compound Poisson process, i.e.

$$X_t = \sum_{j=1}^{N(t)} Y_j,$$

where $N(t)$ is a Poisson process with intensity λ and $\{Y_j\}$ i.i.d. and independent of $N(t)$. Let us consider the stochastic equation with constants μ and σ

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + dX_t),$$

where S_{t-} is left continuous version of S_t.

Theorem

The process $S_t e^{-rt}, t \geq 0$ is a martingale iff $\mu + \lambda E Y_1 = r$.

If $Y_j \geq -1$ then the solution of the stochastic equation is given by

$$S_t = e^{\mu t} e^{\sigma W_t - 1/2\sigma^2 t} exp(\sum_{j=1}^{N(t)} \ln(1 + Y_j)).$$