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An orthogonality relation for the Whittaker functions of the second kind of imaginary order

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An orthogonality relation for the Whittaker functions of the second kind of imaginary order, $W_{\kappa,\mathrm{i}\mu}(x)$, with $\mu \in \mathbb{R}$, is investigated. The integral $\int_0^\infty \mathrm{d}x\, x^{-2} W_{\kappa,\mathrm{i}\mu}(x) W_{\kappa,\mathrm{i}\mu'}(x)$ is shown to be proportional to the sum $\delta(\mu-\mu') + \delta(\mu+\mu')$, where $\delta(\mu \pm \mu')$ is the Dirac delta distribution. The proportionality factor is found to be $\pi^2/[\mu \sinh(2\pi\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)]$. For $\kappa = 0$, the derived formula reduces to the orthogonality relation for the Macdonald functions of imaginary order, discussed recently in the literature.

Keywords: Whittaker functions; orthogonal functions; Dirac delta distribution

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1. Introduction

The Whittaker functions $M_{\kappa,\lambda}(x)$ and $W_{\kappa,\lambda}(x)$ [2,8], closely related to the confluent hypergeometric functions, play an important role in various branches of applied mathematics and theoretical physics, for instance, in fluid mechanics, scalar and electromagnetic diffraction theory or atomic structure theory. This justifies the continuous effort in studying properties of these functions and in gathering information about them. It is the purpose of this brief paper to contribute to the knowledge about the Whittaker function of the second kind.

The following double-integral formula:

$$f(\mu) = \frac{\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)}{\pi^2} \int_0^\infty \mathrm{d}x \frac{W_{\kappa,\mathrm{i}\mu}(x)}{x^2} \times \int_0^\infty \mathrm{d}\mu'\, \mu' \sinh(2\pi\mu') W_{\kappa,\mathrm{i}\mu'}(x) f(\mu') \quad (\mu > 0), \tag{1.1}$$

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valid under certain restrictions imposed on $f(\mu)$, was obtained by Wimp [9] as a particular case of a more general relation involving the Meijer's G-function. From it, one may infer that the Whittaker functions of imaginary order, $W_{\kappa,\mathrm{i}\mu}(x)$ and $W_{\kappa,\mathrm{i}\mu'}(x)$, with $\mu, \mu' > 0$, are orthogonal on the positive real semi-axis with the weight x^{-2} in the sense of

$$\int_0^\infty \mathrm{d}x \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{\pi^2}{\mu\sinh(2\pi\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)} \times \delta(\mu-\mu') \quad (\mu, \mu' > 0), \tag{1.2}$$

where $\delta(\mu - \mu')$ is the Dirac delta function. Evidently, Equation (1.2) is to be understood in the distributional sense. It is of fundamental importance for the theory of integral transforms with kernels involving the Whittaker's W-function, which in recent years attracted interest of mathematicians [6,11] and physicists [1] (for a broader context, see [10]).

Of course, derivation of properties of the Whittaker's W-function from those of the more general G-function is a perfectly valid procedure. However, it is neither simple nor economical, as it requires a good command of the rather complicated theory of Meijer's function. In this context, it seems natural to look for an alternative, direct method of derivation of relation (1.2). Such a method, making use of basic properties of the Whittaker function and elementary facts from the theory of distributions only, is presented in this paper. More precisely, in Section 3, we shall arrive at the relation

$$\int_0^\infty \mathrm{d}x \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{\pi^2}{\mu\sinh(2\pi\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)} \times [\delta(\mu-\mu')+\delta(\mu+\mu')] \quad (\mu, \mu' \in \mathbb{R}), \tag{1.3}$$

which is slightly more general than that in Equation (1.2) and reduces to the latter for $\mu, \mu' > 0$.

Throughout the rest of the work, unless otherwise stated, it is assumed that $\kappa \in \mathbb{C}$, $\mu, \mu' \in \mathbb{R}$ and $x \geqslant 0$.

2. Summary of relevant properties of the Whittaker functions of the second kind of imaginary order

Below we shall list these properties of the Whittaker functions of the second kind of imaginary order, $W_{\kappa,\mathrm{i}\mu}(x)$, which will prove to be helpful in Section 3 for the derivation of relation (1.3). The formulas presented below have been extracted, with slight modifications whenever necessary, from the invaluable collection by Magnus *et al.* [3].

The function $W_{\kappa,\mathrm{i}\mu}(x)$ satisfies the Whittaker differential equation

$$\frac{\mathrm{d}^2F(x)}{\mathrm{d}x^2} + \left(\frac{\mu^2+1/4}{x^2} + \frac{\kappa}{x} - \frac{1}{4}\right)F(x) = 0. \tag{2.1}$$

The pair of the Whittaker functions of the first kind

$$M_{\kappa,\pm\mathrm{i}\mu}(x) = x^{1/2\pm\mathrm{i}\mu}\mathrm{e}^{-x/2}{}_1F_1\left(\frac{1}{2}-\kappa\pm\mathrm{i}\mu; 1\pm2\mathrm{i}\mu; x\right) \tag{2.2}$$

also solves Equation (2.1). The functions $M_{\kappa,\pm\mathrm{i}\mu}(x)$ and $W_{\kappa,\mathrm{i}\mu}(x)$ are not independent but are related through

$$W_{\kappa,\mathrm{i}\mu}(x) = \frac{\Gamma(2\mathrm{i}\mu)}{\Gamma(1/2-\kappa+\mathrm{i}\mu)}M_{\kappa,-\mathrm{i}\mu}(x) + \frac{\Gamma(-2\mathrm{i}\mu)}{\Gamma(1/2-\kappa-\mathrm{i}\mu)}M_{\kappa,\mathrm{i}\mu}(x). \tag{2.3}$$

Equation (2.3) may serve as a definition of $W_{\kappa,\mathrm{i}\mu}(x)$ in terms of $M_{\kappa,\pm\mathrm{i}\mu}(x)$.

For large positive values of x the function $W_{\kappa,\mathrm{i}\mu}(x)$ has the asymptotic representation

$$W_{\kappa,\mathrm{i}\mu}(x) \overset{x\to\infty}{\sim} x^{\kappa}\mathrm{e}^{-x/2}{}_2F_0\left(\frac{1}{2}-\kappa+\mathrm{i}\mu, \frac{1}{2}-\kappa-\mathrm{i}\mu; ; -x^{-1}\right). \tag{2.4}$$

From Equations (2.3) and (2.2), one finds that for small positive values of x the function $W_{\kappa,\mathrm{i}\mu}(x)$ behaves as

$$W_{\kappa,\mathrm{i}\mu}(x) \overset{x\to 0+}{\sim} x^{1/2}[A_{\kappa,\mathrm{i}\mu}\cos(-\mu\ln x) + B_{\kappa,\mathrm{i}\mu}\sin(-\mu\ln x)][1+O(x^{-1})], \tag{2.5}$$

where

$$A_{\kappa,\mathrm{i}\mu} = \frac{\Gamma(-2\mathrm{i}\mu)}{\Gamma(1/2-\kappa-\mathrm{i}\mu)} + \frac{\Gamma(2\mathrm{i}\mu)}{\Gamma(1/2-\kappa+\mathrm{i}\mu)}, \tag{2.6}$$

$$\mathrm{i}B_{\kappa,\mathrm{i}\mu} = \frac{\Gamma(-2\mathrm{i}\mu)}{\Gamma(1/2-\kappa-\mathrm{i}\mu)} - \frac{\Gamma(2\mathrm{i}\mu)}{\Gamma(1/2-\kappa+\mathrm{i}\mu)}. \tag{2.7}$$

3. Orthogonality relation for the Whittaker functions of the second kind of imaginary order

Consider two Whittaker functions $W_{\kappa,\mathrm{i}\mu}(x)$ and $W_{\kappa,\mathrm{i}\mu'}(x)$. According to what has been said in Section 2, they obey the differential identities

$$\frac{\mathrm{d}^2 W_{\kappa,\mathrm{i}\mu}(x)}{\mathrm{d}x^2} + \left(\frac{\mu^2+1/4}{x^2} + \frac{\kappa}{x} - \frac{1}{4}\right)W_{\kappa,\mathrm{i}\mu}(x) = 0 \tag{3.1}$$

and

$$\frac{\mathrm{d}^2 W_{\kappa,\mathrm{i}\mu'}(x)}{\mathrm{d}x^2} + \left(\frac{\mu'^2+1/4}{x^2} + \frac{\kappa}{x} - \frac{1}{4}\right)W_{\kappa,\mathrm{i}\mu'}(x) = 0. \tag{3.2}$$

We premultiply Equation (3.1) by $W_{\kappa,\mathrm{i}\mu'}(x)$, Equation (3.2) by $W_{\kappa,\mathrm{i}\mu}(x)$, subtract and integrate the result over x from some $\xi > 0$ to ∞. After obvious movements, this gives

$$(\mu^2-\mu'^2)\int_{\xi}^{\infty}\mathrm{d}x\,\frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \int_{\xi}^{\infty}\mathrm{d}x\left[W_{\kappa,\mathrm{i}\mu}(x)\frac{\mathrm{d}^2 W_{\kappa,\mathrm{i}\mu'}(x)}{\mathrm{d}x^2} - W_{\kappa,\mathrm{i}\mu'}(x)\frac{\mathrm{d}^2 W_{\kappa,\mathrm{i}\mu}(x)}{\mathrm{d}x^2}\right]. \tag{3.3}$$

Integrating the right-hand side by parts, we obtain

$$(\mu^2-\mu'^2)\int_{\xi}^{\infty}\mathrm{d}x\,\frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \left[W_{\kappa,\mathrm{i}\mu}(x)\frac{\mathrm{d}W_{\kappa,\mathrm{i}\mu'}(x)}{\mathrm{d}x} - W_{\kappa,\mathrm{i}\mu'}(x)\frac{\mathrm{d}W_{\kappa,\mathrm{i}\mu}(x)}{\mathrm{d}x}\right]_{x=\xi}^{\infty}. \tag{3.4}$$

Hence, after making use of the asymptotic property (2.4), it follows that

$$\int_{0}^{\infty}\mathrm{d}x\,\frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = -\lim_{\xi\to 0+}\frac{W_{\kappa,\mathrm{i}\mu}(\xi)(\mathrm{d}W_{\kappa,\mathrm{i}\mu'}(\xi)/\mathrm{d}\xi) - W_{\kappa,\mathrm{i}\mu'}(\xi)(\mathrm{d}W_{\kappa,\mathrm{i}\mu}(\xi)/\mathrm{d}\xi)}{\mu^2-\mu'^2}. \tag{3.5}$$

To evaluate the limit on the right-hand side, we again exploit an asymptotic property of the Whittaker function, this time the one in Equation (2.5). This yields

$$\begin{aligned}\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} &= \lim_{\xi\to 0+}\Bigg[\frac{\mu A_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'} - \mu' B_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'}}{\mu^2-\mu'^2}\sin(-\mu\ln\xi)\sin(-\mu'\ln\xi)\\ &\quad + \frac{\mu A_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'} + \mu' B_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'}}{\mu^2-\mu'^2}\sin(-\mu\ln\xi)\cos(-\mu'\ln\xi)\\ &\quad - \frac{\mu B_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'} + \mu' A_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'}}{\mu^2-\mu'^2}\cos(-\mu\ln\xi)\sin(-\mu'\ln\xi)\\ &\quad - \frac{\mu B_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'} - \mu' A_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'}}{\mu^2-\mu'^2}\cos(-\mu\ln\xi)\cos(-\mu'\ln\xi)\Bigg].\end{aligned} \tag{3.6}$$

With a little trigonometry, this may be transformed into

$$\begin{aligned}\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} &= \frac{1}{2}\lim_{\xi\to 0+}\Bigg\{(A_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'} + B_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'})\frac{\sin[-(\mu-\mu')\ln\xi]}{\mu-\mu'}\\ &\quad + (A_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'} - B_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'})\frac{\sin[-(\mu+\mu')\ln\xi]}{\mu+\mu'}\\ &\quad + \frac{A_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'} - B_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'}}{\mu-\mu'}\cos[-(\mu-\mu')\ln\xi]\\ &\quad - \frac{A_{\kappa,\mathrm{i}\mu}B_{\kappa,\mathrm{i}\mu'} + B_{\kappa,\mathrm{i}\mu}A_{\kappa,\mathrm{i}\mu'}}{\mu+\mu'}\cos[-(\mu+\mu')\ln\xi]\Bigg\}.\end{aligned} \tag{3.7}$$

It is known from the elementary theory of distributions [5, Section 4.5] that

$$\lim_{a\to\infty}\frac{\sin(ax)}{\pi x} = \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^{a}\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\eta x} = \delta(x) \tag{3.8}$$

and also that (in the distributional sense)

$$\lim_{a\to\infty}\cos(ax) = 0 \tag{3.9}$$

(the latter is the corollary from the Riemann–Lebesgue lemma). Since for $\xi \to 0+$ one has $-\ln\xi \to \infty$, application of the above distributional relations and Equations (2.6) and (2.7) to Equation (3.7) casts the latter into

$$\begin{aligned}&\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2}\\ &\quad = \pi\left[\frac{\Gamma(2\mathrm{i}\mu)\Gamma(-2\mathrm{i}\mu')}{\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu')} + \frac{\Gamma(-2\mathrm{i}\mu)\Gamma(2\mathrm{i}\mu')}{\Gamma(1/2-\kappa-\mathrm{i}\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu')}\right]\delta(\mu-\mu')\\ &\qquad + \pi\left[\frac{\Gamma(2\mathrm{i}\mu)\Gamma(2\mathrm{i}\mu')}{\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu')} + \frac{\Gamma(-2\mathrm{i}\mu)\Gamma(-2\mathrm{i}\mu')}{\Gamma(1/2-\kappa-\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu')}\right]\\ &\qquad \times \delta(\mu+\mu') \quad (\mu,\mu'\in\mathbb{R}).\end{aligned} \tag{3.10}$$

This is the symmetric form of the orthogonality relation for the Whittaker functions of the second kind of imaginary order. Upon making use of the following known property of the Dirac delta [5, Section 4.4]:

$$g(\mu')\delta(\mu \mp \mu') = g(\pm\mu)\delta(\mu \mp \mu'), \tag{3.11}$$

relation (3.10) may be rewritten compactly, although unsymmetrically, as

$$\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{2\pi|\Gamma(2\mathrm{i}\mu)|^2}{\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)}[\delta(\mu-\mu')+\delta(\mu+\mu')]. \tag{3.12}$$

This may be still simplified. Use of the formula

$$|\Gamma(2\mathrm{i}\mu)| = \sqrt{\frac{\pi}{2\mu\sinh(2\pi\mu)}} \tag{3.13}$$

transforms Equation (3.12) into

$$\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{\pi^2}{\mu\sinh(2\pi\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)} \times [\delta(\mu-\mu')+\delta(\mu+\mu')] \quad (\mu,\mu'\in\mathbb{R}). \tag{3.14}$$

An alternative form of the above relation is obtained if one rewrites the right-hand side with the use of the identity

$$\delta(\mu^2-\mu'^2) = \frac{1}{2|\mu|}[\delta(\mu-\mu')+\delta(\mu+\mu')]. \tag{3.15}$$

This yields

$$\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{2\pi^2}{\sinh(2\pi|\mu|)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)}\,\delta(\mu^2-\mu'^2) \quad (\mu,\mu'\in\mathbb{R}). \tag{3.16}$$

If $\mu, \mu' > 0$, then $\mu+\mu' \neq 0$ and consequently in the distributional sense it holds that

$$\delta(\mu+\mu') = 0 \quad (\mu,\mu' > 0). \tag{3.17}$$

In this case, relation (3.14) reduces to

$$\int_0^\infty \mathrm{d}x\, \frac{W_{\kappa,\mathrm{i}\mu}(x)W_{\kappa,\mathrm{i}\mu'}(x)}{x^2} = \frac{\pi^2}{\mu\sinh(2\pi\mu)\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)}\,\delta(\mu-\mu') \quad (\mu,\mu' > 0). \tag{3.18}$$

Concluding, we observe that if κ is an integer or half-integer, the product $\Gamma(1/2-\kappa+\mathrm{i}\mu)\Gamma(1/2-\kappa-\mathrm{i}\mu)$ appearing in relations (3.14), (3.16) and (3.18) may be expressed in terms of elementary functions. Of particular interest is the case $\kappa = 0$, since then one has

$$W_{0,\mathrm{i}\mu}(x) = \sqrt{\frac{x}{\pi}} K_{\mathrm{i}\mu}\left(\frac{1}{2}x\right), \tag{3.19}$$

where $K_{\mathrm{i}\mu}(x)$ is the Macdonald function of imaginary order used as a kernel in the Kontorovich–Lebedev transform. Exploiting the identity

$$\left|\Gamma\left(\frac{1}{2} + \mathrm{i}\mu\right)\right| = \sqrt{\frac{\pi}{\cosh(\pi\mu)}}, \tag{3.20}$$

we find that the orthogonality relations (3.14) and (3.18) go over into

$$\int_0^\infty \mathrm{d}x \frac{K_{\mathrm{i}\mu}(x)K_{\mathrm{i}\mu'}(x)}{x} = \frac{\pi^2}{2\mu\sinh(\pi\mu)}[\delta(\mu - \mu') + \delta(\mu + \mu')] \quad (\mu, \mu' \in \mathbb{R}) \tag{3.21}$$

and

$$\int_0^\infty \mathrm{d}x \, \frac{K_{\mathrm{i}\mu}(x)K_{\mathrm{i}\mu'}(x)}{x} = \frac{\pi^2}{2\mu\sinh(\pi\mu)}\,\delta(\mu - \mu') \qquad (\mu, \mu' > 0), \tag{3.22}$$

respectively. These relations have been derived by Szmytkowski and Bielski [7] in the manner analogous to the one which has led us above to Equations (3.14) and (3.18). Somewhat earlier, Yakubovich [12] and Passian *et al.* [4] proved the validity of relation (3.22) exploiting more involved mathematical techniques.

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