

Closed form of the generalized Green's function for the Helmholtz operator on the two-dimensional unit sphere

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The closed representation of the generalized (known also as reduced or modified) Green's function for the Helmholtz partial differential operator on the surface of the two-dimensional unit sphere is derived. In its compact form, the derived formula contains a Legendre polynomial and a derivative of the Legendre function of the first kind with respect to its index. An explicit expression for that derivative is found and used to obtain an expanded (and potentially more suitable in applications) form of the generalized Green's function for the operator in question. The related problem of constructing the closed form of the generalized Green's function for the Legendre ordinary differential operator on the segment $-1 < x < 1$, with the boundary conditions of finiteness at $x = \pm 1$, is also solved. © 2006 American Institute of Physics. [DOI: [10.1063/1.2203430](https://doi.org/10.1063/1.2203430)]

I. INTRODUCTION

Let S^2 be the surface of the unit sphere in R^3 , parametrized by the unit radius vector \mathbf{n} . The spatial orientation of the vector \mathbf{n} (hence, also the location of a point on S^2 for which \mathbf{n} is the radius vector) is uniquely determined by fixing the polar angle $0 \leq \theta \leq \pi$ and the azimuthal angle $0 \leq \varphi < 2\pi$ in some spherical system of coordinates, with the origin of the latter located at the center of S^2 .

Some models of propagation of time-harmonic waves on spherical surfaces used in mathematical geophysics (cf., e.g., the recent work of Yoshizawa and Kennett²⁹) lead to the inhomogeneous scalar Helmholtz equation on S^2 :

$$[\nabla_{\mathbf{n}}^2 + \lambda(\lambda + 1)]\Psi(\lambda; \mathbf{n}) = \Phi(\mathbf{n}). \quad (1.1)$$

In this equation

$$\nabla_{\mathbf{n}}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \quad (1.2)$$

is the angular part of the Laplace operator (the spherical Laplacian) and $\Phi(\mathbf{n})$ is a source term. For the sake of later convenience, the square of the (in general complex) propagation constant has been written in the form $\lambda(\lambda+1)$, where $\lambda \in \mathbb{C}$. Equation (1.1) is to be solved subject to some boundary and/or regularity conditions imposed on $\Psi(\lambda; \mathbf{n})$. If the wave described by Eq. (1.1) propagates over the entire spherical surface, which will be implicit throughout the rest of this paper, these conditions are the single valuedness and finiteness of $\Psi(\lambda; \mathbf{n})$.

With present-day computers and software (see, e.g., the paper by Adams and Swartrauber¹), Eq. (1.1), with the aforementioned regularity conditions, may be solved numerically on S^2 for a

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practically arbitrary form of the source function $\Phi(\mathbf{n})$. On the other hand, it appears that the formal analytical solution to Eq. (1.1) is also available; except for the case when $\lambda(\lambda+1)$ is an eigenvalue of $-\nabla_{\mathbf{n}}^2$ on S^2 , it is given by

$$\Psi(\lambda; \mathbf{n}) = \oint_{S^2} d^2\mathbf{n}' G(\lambda; \mathbf{n}, \mathbf{n}') \Phi(\mathbf{n}'), \quad (1.3)$$

where $G(\lambda; \mathbf{n}, \mathbf{n}')$ is the Green's function for the Helmholtz operator

$$\mathcal{H}(\lambda; \mathbf{n}) = \nabla_{\mathbf{n}}^2 + \lambda(\lambda + 1) \quad (1.4)$$

on S^2 . Following the general theory,^{8,12,13,15,17,20,23,25,27,28} $G(\lambda; \mathbf{n}, \mathbf{n}')$ is defined as a solution to the inhomogeneous partial differential equation (with \mathbf{n}' fixed)

$$[\nabla_{\mathbf{n}}^2 + \lambda(\lambda + 1)]G(\lambda; \mathbf{n}, \mathbf{n}') = \delta^{(2)}(\mathbf{n} - \mathbf{n}'), \quad (1.5)$$

subject to the constraints of single-valuedness and finiteness (with the latter relaxed at the point $\mathbf{n} = \mathbf{n}'$). In Eq. (1.5) the inhomogeneity $\delta^{(2)}(\mathbf{n} - \mathbf{n}')$ is the Dirac delta function on S^2 . The closed form of the Green's function $G(\lambda; \mathbf{n}, \mathbf{n}')$ is known^{2-7,9,10,24,26} to be

$$G(\lambda; \mathbf{n}, \mathbf{n}') = \frac{1}{4 \sin(\pi\lambda)} P_{\lambda}(-\mathbf{n} \cdot \mathbf{n}') \quad (\lambda \notin \mathbb{Z}), \quad (1.6)$$

where

$$P_{\lambda}(x) = {}_2F_1\left(-\lambda, \lambda + 1; 1; \frac{1-x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-\lambda)_n (\lambda + 1)_n}{(n!)^2} \left(\frac{1-x}{2}\right)^n \quad (-1 \leq x \leq 1) \quad (1.7)$$

is the Legendre function of the first kind,^{16,22} with

$$(\zeta)_n = \frac{\Gamma(\zeta + n)}{\Gamma(\zeta)} \quad (n \in \mathbb{N}) \quad (1.8)$$

denoting the Pochhammer symbol.²²

Equation (1.3) fails to represent a solution of Eq. (1.1) on S^2 when $\lambda \in \mathbb{Z}$, i.e., if it holds that

$$\lambda(\lambda + 1) = L(L + 1) \quad (L \in \mathbb{N}). \quad (1.9)$$

This corresponds to the situation when the operator (1.4) has a null eigenvalue in its spectrum. It is known from the general theory of linear inhomogeneous equations²⁰ that in such a case a solution to Eq. (1.1) exists only if the source term is orthogonal to the null space of the operator $\mathcal{H}(L; \mathbf{n})$, spanned by $2L+1$ complex spherical harmonics

$$Y_{LM}(\mathbf{n}) = \frac{(-)^{L+M}}{2^L L!} \sqrt{\frac{2L+1}{4\pi} \frac{(L-M)!}{(L+M)!}} \sin^M \theta \frac{d^{L+M} \sin^{2L} \theta}{d \cos^{L+M} \theta} e^{iM\varphi}, \quad (1.10)$$

with $M \in \{0, \pm 1, \dots, \pm L\}$ [the phase choice in Eq. (1.10) conforms to the widely accepted Condon and Shortley¹¹ convention], i.e., when

$$\oint_{S^2} d^2\mathbf{n} Y_{LM}^*(\mathbf{n}) \Phi(\mathbf{n}) = 0 \quad (M \in \{0, \pm 1, \dots, \pm L\}). \quad (1.11)$$

If the constraints (1.9) and (1.11) hold simultaneously, a (nonunique) solution to the spherical Helmholtz equation (1.1) is formally given by

$$\Psi_L(\mathbf{n}) = \sum_{M=-L}^L a_{LM} Y_{LM}(\mathbf{n}) + \oint_{S^2} d^2\mathbf{n}' \bar{G}_L(\mathbf{n}, \mathbf{n}') \Phi(\mathbf{n}'), \quad (1.12)$$

where $\{a_{LM}\}$, ($M \in \{0, \pm 1, \dots, \pm L\}$), are arbitrary constants, while $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ is a generalized (known also as reduced or modified) Green's function for the Helmholtz operator $\mathcal{H}(L; \mathbf{n})$. This function is defined as this particular solution to the inhomogeneous partial differential equation

$$[\nabla_{\mathbf{n}}^2 + L(L+1)]\bar{G}_L(\mathbf{n}, \mathbf{n}') = \delta^{(2)}(\mathbf{n} - \mathbf{n}') - \sum_{M=-L}^L Y_{LM}(\mathbf{n}) Y_{LM}^*(\mathbf{n}') \quad (L \in \mathbb{N}), \quad (1.13)$$

which is single-valued and finite on S^2 (except at the point $\mathbf{n} = \mathbf{n}'$), being, in addition, orthogonal to the null space of the operator $\mathcal{H}(L; \mathbf{n})$:

$$\oint_{S^2} d^2\mathbf{n}' Y_{LM}^*(\mathbf{n}') \bar{G}_L(\mathbf{n}, \mathbf{n}') = 0 \quad (M \in \{0, \pm 1, \dots, \pm L\}). \quad (1.14)$$

In contrary to the case of $G(\lambda; \mathbf{n}, \mathbf{n}')$, despite performing the extensive search through the mathematical and physical literature, we have found no studies on the *closed* form of $\bar{G}_L(\mathbf{n}, \mathbf{n}')$, except for the particular case $L=0$ (cf. Refs. 12, 14, 18, and 19). It is therefore the purpose of this work to fill in this gap by presenting the construction of the closed representation of the generalized Green's function for the Helmholtz operator (1.4), constrained by Eq. (1.9), on the unit sphere S^2 .

The structure of the paper is as follows. First, in Sec. II we show that $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ may be expressed compactly in terms of the derivative $[\partial P_\lambda(-\mathbf{n} \cdot \mathbf{n}') / \partial \lambda]_{\lambda=L}$. Then, in Sec. III we find the explicit representation of $[\partial P_\lambda(x) / \partial \lambda]_{\lambda=L}$, the result which, apart from being important in the context of the present work, seems to be also of interest for itself. Finally, in Sec. IV we combine the findings of Secs. II and III, arriving at the sought explicit closed form of $\bar{G}_L(\mathbf{n}, \mathbf{n}')$. We also show that in the particular case $L=0$ the known result for the spherical Laplacian^{14,19} is recovered. In addition, we provide the inhomogeneous three-term recurrence relation satisfied by $\bar{G}_L(\mathbf{n}, \mathbf{n}')$. The intermediate result obtained in Sec. III is of wider applicability than the subject of this work might suggest. We illustrate this in the Appendix, where the closed form of the generalized Green's function for the one-dimensional Legendre operator

$$\mathcal{L}(L; x) = \frac{d}{dx}(1-x^2) \frac{d}{dx} + L(L+1) \quad (L \in \mathbb{N}) \quad (1.15)$$

on the interval $-1 < x < 1$, with the boundary conditions of finiteness at $x = \pm 1$, is constructed.

II. COMPACT CLOSED FORM OF THE GENERALIZED GREEN'S FUNCTION FOR THE HELMHOLTZ OPERATOR ON S^2

From the general theory of Green's functions (cf. the relevant references cited in Sec. I) it follows that one may construct the generalized spherical Helmholtz Green's function $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ in the form of the so-called spectral series

$$\bar{G}_L(\mathbf{n}, \mathbf{n}') = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}')}{L(L+1) - l(l+1)}. \quad (2.1)$$

($l \neq L$)

Comparing this with the spectral series representation of the Green's function $G(\lambda; \mathbf{n}, \mathbf{n}')$, known to be

$$G(\lambda; \mathbf{n}, \mathbf{n}') = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}')}{\lambda(\lambda+1) - l(l+1)}, \quad (2.2)$$

it may be inferred that the functions $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ and $G(\lambda; \mathbf{n}, \mathbf{n}')$ are related through

$$\bar{G}_L(\mathbf{n}, \mathbf{n}') = \lim_{\lambda(\lambda+1) \rightarrow L(L+1)} \frac{\partial}{\partial[\lambda(\lambda+1)]} \{[\lambda(\lambda+1) - L(L+1)] G(\lambda; \mathbf{n}, \mathbf{n}')\}. \quad (2.3)$$

This, after exploiting the result (1.6), may be rewritten as

$$\bar{G}_L(\mathbf{n}, \mathbf{n}') = \frac{1}{4(2L+1)} \lim_{\lambda \rightarrow L} \frac{\partial}{\partial \lambda} \frac{(\lambda-L)(\lambda+L+1)}{\sin(\pi\lambda)} P_{\lambda}(-\mathbf{n} \cdot \mathbf{n}'). \quad (2.4)$$

The limit on the right-hand side of the above equation may be evaluated by using the l'Hospital rule. After exploiting the fact that for a non-negative integer index the Legendre function (1.7) degenerates to the Legendre polynomial

$$P_L(x) = \frac{1}{2^L L!} \frac{d^L(x^2-1)^L}{dx^L} \quad (L \in \mathbb{N}), \quad (2.5)$$

and making use of the following reflection property of the Legendre polynomials:

$$P_L(-x) = (-)^L P_L(x) \quad (L \in \mathbb{N}), \quad (2.6)$$

one eventually finds that the closed form of $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ is

$$\bar{G}_L(\mathbf{n}, \mathbf{n}') = \left. \frac{(-)^L \partial P_{\lambda}(-\mathbf{n} \cdot \mathbf{n}')}{4\pi \partial \lambda} \right|_{\lambda=L} + \frac{1}{4\pi} \frac{P_L(\mathbf{n} \cdot \mathbf{n}')}{2L+1}. \quad (2.7)$$

From the purely theoretical point of view, arriving at Eq. (2.7) completes the task of determining the closed form of $\bar{G}_L(\mathbf{n}, \mathbf{n}')$. However, in most of the actual applications formula (2.7) will be unhandy, if not useless. To find a representation of $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ which is suitable for practical purposes, the derivative $[\partial P_{\lambda}(x)/\partial \lambda]_{\lambda=L}$ must be evaluated explicitly. We shall be concerned with the latter problem in the following section.

III. EVALUATION OF $[\partial P_{\lambda}(x)/\partial \lambda]_{\lambda=L}$

Throughout the whole section, it will be implicit that L is an arbitrary non-negative integer and that $-1 \leq x \leq 1$.

Consider the definition (1.7). Differentiating it with respect to λ , after making use of the following differential property of the Pochhammer symbol:

$$\frac{d(\zeta)_n}{d\zeta} = [\psi(\zeta+n) - \psi(\zeta)](\zeta)_n, \quad (3.1)$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)} \frac{d\Gamma(\zeta)}{d\zeta} \quad (3.2)$$

is the digamma function,^{16,22} one obtains

$$\frac{\partial P_\lambda(x)}{\partial \lambda} = \sum_{n=1}^{\infty} \frac{(-\lambda)_n(\lambda+1)_n}{(n!)^2} [\psi(\lambda+1+n) - \psi(\lambda+1) + \psi(-\lambda) - \psi(-\lambda+n)] \left(\frac{1-x}{2}\right)^n. \quad (3.3)$$

It is evident from Eq. (3.3) that the function $\partial P_\lambda(x)/\partial \lambda$ obeys

$$\left. \frac{\partial P_{\lambda'}(x)}{\partial \lambda'} \right|_{\lambda'=-\lambda-1} = - \left. \frac{\partial P_{\lambda'}(x)}{\partial \lambda'} \right|_{\lambda'=\lambda}. \quad (3.4)$$

The right-hand side of Eq. (3.3) may be simplified. On exploiting the known relation^{16,22}

$$\psi(\zeta) = \psi(1-\zeta) - \pi \cot(\pi\zeta) \quad (3.5)$$

one finds

$$-\psi(\lambda+1) + \psi(-\lambda) - \psi(-\lambda+n) = -\psi(\lambda+1-n). \quad (3.6)$$

Hence, it follows that

$$\frac{\partial P_\lambda(x)}{\partial \lambda} = \sum_{n=1}^{\infty} \frac{(-\lambda)_n(\lambda+1)_n}{(n!)^2} [\psi(\lambda+1+n) - \psi(\lambda+1-n)] \left(\frac{1-x}{2}\right)^n. \quad (3.7)$$

Equation (3.7) does not look simple. However, at this stage we may take advantage of the fact that for the purposes of the present work the derivative $\partial P_\lambda(x)/\partial \lambda$ is to be evaluated for $\lambda=L \in \mathbb{N}$ only.

At first, consider what happens if $\lambda=0$. Since

$$\lim_{\lambda \rightarrow 0} \frac{\psi(\lambda+1+n)}{\Gamma(-\lambda)} = 0 \quad (n \in \mathbb{N}) \quad (3.8)$$

and

$$\lim_{\lambda \rightarrow 0} \frac{\psi(\lambda+1-n)}{\Gamma(-\lambda)} = 1 \quad (n \in \mathbb{N} \setminus \{0\}), \quad (3.9)$$

in this particular case we arrive at the known result^{16,22}

$$\left. \frac{\partial P_\lambda(x)}{\partial \lambda} \right|_{\lambda=0} = - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-x}{2}\right)^n = \ln \frac{1+x}{2}. \quad (3.10)$$

Next, the Legendre function (1.7) is known^{16,22} to satisfy the three-term homogeneous recurrence relation

$$(\lambda+1)P_{\lambda+1}(x) - (2\lambda+1)xP_\lambda(x) + \lambda P_{\lambda-1}(x) = 0. \quad (3.11)$$

Differentiating this relation with respect to λ yields the three-term inhomogeneous recurrence relation for the function (3.7)

$$(\lambda+1) \frac{\partial P_{\lambda+1}(x)}{\partial \lambda} - (2\lambda+1)x \frac{\partial P_\lambda(x)}{\partial \lambda} + \lambda \frac{\partial P_{\lambda-1}(x)}{\partial \lambda} = -P_{\lambda+1}(x) + 2xP_\lambda(x) - P_{\lambda-1}(x), \quad (3.12)$$

which, upon exploiting again Eq. (3.11), may be rewritten in the form

$$(\lambda + 1) \frac{\partial P_{\lambda+1}(x)}{\partial \lambda} - (2\lambda + 1)x \frac{\partial P_{\lambda}(x)}{\partial \lambda} + \lambda \frac{\partial P_{\lambda-1}(x)}{\partial \lambda} = \frac{1}{2\lambda + 1} [P_{\lambda+1}(x) - P_{\lambda-1}(x)]. \quad (3.13)$$

If $\lambda = L \in \mathbb{N}$, the recurrence (3.13) becomes

$$(L + 1) \left. \frac{\partial P_{\lambda}(x)}{\partial \lambda} \right|_{\lambda=L+1} - (2L + 1)x \left. \frac{\partial P_{\lambda}(x)}{\partial \lambda} \right|_{\lambda=L} + L \left. \frac{\partial P_{\lambda}(x)}{\partial \lambda} \right|_{\lambda=L-1} = \frac{1}{2L + 1} [P_{L+1}(x) - P_{L-1}(x)]. \quad (3.14)$$

Equation (3.14) is to be solved with formula (3.10) taken as an initial condition. Now, it is known from the general theory of linear difference equations²¹ that the solution to the problem (3.14) and (3.10) may be sought in the form

$$\left. \frac{\partial P_{\lambda}(x)}{\partial \lambda} \right|_{\lambda=L} = F_L(x) + W_L(x), \quad (3.15)$$

where $F_L(x)$ solves

$$(L + 1)F_{L+1}(x) - (2L + 1)x F_L(x) + L F_{L-1}(x) = 0 \quad (3.16)$$

subject to the inhomogeneous initial condition

$$F_0(x) = \ln \frac{1+x}{2}, \quad (3.17)$$

while $W_L(x)$ satisfies

$$(L + 1)W_{L+1}(x) - (2L + 1)x W_L(x) + L W_{L-1}(x) = \frac{1}{2L + 1} [P_{L+1}(x) - P_{L-1}(x)] \quad (3.18)$$

subject to the homogeneous initial condition

$$W_0(x) = 0. \quad (3.19)$$

Equations (3.18) and (3.19) imply that $W_L(x)$ is a polynomial in x , of degree L , such that

$$W_L(1) = 0. \quad (3.20)$$

We shall make use of this observation shortly.

The solution to the problem constituted by Eqs. (3.16) and (3.17) is immediately found to be

$$F_L(x) = P_L(x) \ln \frac{1+x}{2}. \quad (3.21)$$

To find $W_L(x)$, we observe that the derivative $\partial P_{\lambda}(x)/\partial \lambda$ satisfies the inhomogeneous differential equation

$$\left[\frac{d}{dx}(1-x^2) \frac{d}{dx} + \lambda(\lambda+1) \right] \frac{\partial P_{\lambda}(x)}{\partial \lambda} = -(2\lambda+1)P_{\lambda}(x), \quad (3.22)$$

obtainable from the Legendre identity

$$\left[\frac{d}{dx}(1-x^2) \frac{d}{dx} + \lambda(\lambda+1) \right] P_{\lambda}(x) = 0 \quad (3.23)$$

by its differentiation with respect to λ . On combining Eqs. (3.22), (3.15), and (3.21), and after making use of the known relationship²²

$$x \frac{dP_L(x)}{dx} - LP_L(x) = \frac{dP_{L-1}(x)}{dx}, \quad (3.24)$$

one deduces that $W_L(x)$ satisfies the inhomogeneous differential equation

$$\left[\frac{d}{dx}(1-x^2) \frac{d}{dx} + L(L+1) \right] W_L(x) = 2 \frac{dP_{L-1}(x)}{dx} - 2 \frac{dP_L(x)}{dx}. \quad (3.25)$$

Since $W_L(x)$ is the polynomial, the pertinent boundary conditions supplementing Eq. (3.25) are

$$W_L(x) \text{ bounded for } x \rightarrow \pm 1. \quad (3.26)$$

It is evident that by solving the boundary value problem constituted by Eqs. (3.25) and (3.26) one is able to determine $W_L(x)$ only up to a multiple of the Legendre polynomial $P_L(x)$, since the latter solves the associated homogeneous boundary value problem. With this fact in mind, on applying the machinery of Green's functions, $W_L(x)$ is found to have the form

$$W_L(x) = c_L P_L(x) + 2 \int_{-1}^1 dx' \bar{g}_L(x, x') \left[\frac{dP_{L-1}(x')}{dx'} - \frac{dP_L(x')}{dx'} \right], \quad (3.27)$$

where

$$\bar{g}_L(x, x') = \frac{1}{2} \sum_{\substack{l=0 \\ (l \neq L)}}^{\infty} \frac{2l+1}{(L-l)(L+l+1)} P_l(x) P_l(x') \quad (3.28)$$

is the generalized Green's function (in its spectral form, cf. the Appendix) for the Legendre operator (1.15), while c_L is a coefficient which remains to be determined. The integral on the right-hand side of Eq. (3.27) may be evaluated if one makes use of the known relationship¹⁶

$$\frac{dP_L(x)}{dx} = \sum_{l=0}^{\text{int}[(L-1)/2]} (2L-4l-1) P_{L-2l-1}(x) \quad (3.29)$$

[in Eq. (3.29), and hereafter, it is implicit that if the upper limit of the summation is smaller than the lower one, than the sum is identically zero], which implies that it holds that

$$\frac{dP_{L-1}(x)}{dx} - \frac{dP_L(x)}{dx} = \sum_{l=0}^{L-1} (-)^{L+l} (2l+1) P_l(x). \quad (3.30)$$

On inserting Eqs. (3.28) and (3.30) into Eq. (3.27), after exploiting the orthogonality property^{16,22}

$$\int_{-1}^1 dx P_L(x) P_{L'}(x) = \frac{2}{2L+1} \delta_{LL'}, \quad (3.31)$$

one obtains $W_L(x)$ in the form

$$W_L(x) = c_L P_L(x) + 2 \sum_{l=0}^{L-1} (-)^{L+l} \frac{2l+1}{(L-l)(L+l+1)} P_l(x). \quad (3.32)$$

It remains to determine the coefficient c_L . To this end, in Eq. (3.32) one sets $x=1$ and then exploits the property

$$P_\lambda(1) = 1, \quad (3.33)$$

and also Eq. (3.20), obtaining

$$c_L = -2 \sum_{l=0}^{L-1} (-)^{L+l} \frac{2l+1}{(L-l)(L+l+1)}. \quad (3.34)$$

This leads to the following representation of the polynomial $W_L(x)$:

$$W_L(x) = 2 \sum_{l=0}^{L-1} (-)^{L+l} \frac{2l+1}{(L-l)(L+l+1)} [P_l(x) - P_L(x)]. \quad (3.35)$$

[Observe that, according to the convention introduced below Eq. (3.29), from Eq. (3.35) it follows that $W_0(x)=0$, which is in agreement with Eq. (3.19).]

The results obtained above may be summarized in the following formula for the sought derivative $[\partial P_\lambda(x)/\partial \lambda]_{\lambda=L}$:

$$\left. \frac{\partial P_\lambda(x)}{\partial \lambda} \right|_{\lambda=L} = P_L(x) \ln \frac{1+x}{2} + 2 \sum_{l=0}^{L-1} (-)^{L+l} \frac{2l+1}{(L-l)(L+l+1)} [P_l(x) - P_L(x)]. \quad (3.36)$$

In particular, from Eq. (3.36) it follows that

$$\left. \frac{\partial P_\lambda(x)}{\partial \lambda} \right|_{\lambda=1} = x \ln \frac{1+x}{2} + (x-1), \quad (3.37)$$

$$\left. \frac{\partial P_\lambda(x)}{\partial \lambda} \right|_{\lambda=2} = \frac{1}{2} (3x^2 - 1) \ln \frac{1+x}{2} + \left(\frac{7}{4} x^2 - \frac{3}{2} x - \frac{1}{4} \right), \quad (3.38)$$

$$\left. \frac{\partial P_\lambda(x)}{\partial \lambda} \right|_{\lambda=3} = \frac{1}{2} (5x^3 - 3x) \ln \frac{1+x}{2} + \left(\frac{37}{12} x^3 - \frac{5}{2} x^2 - \frac{5}{4} x + \frac{2}{3} \right). \quad (3.39)$$

We parenthetically observe also that, if needed, the derivatives $[\partial P_\lambda(x)/\partial \lambda]_{\lambda=-L-1}$ may be deduced from Eqs. (3.4) and (3.36).

IV. DISCUSSION

From Eqs. (2.7), (3.36), and (2.6) one deduces that the explicit form of the generalized Green's function for the Helmholtz operator (1.4), with the constraint (1.9), is

$$\begin{aligned} \bar{G}_L(\mathbf{n}, \mathbf{n}') &= \frac{1}{4\pi} P_L(\mathbf{n} \cdot \mathbf{n}') \left[\ln \frac{1 - \mathbf{n} \cdot \mathbf{n}'}{2} - 2 \sum_{l=0}^{L-1} (-)^{L+l} \frac{2l+1}{(L-l)(L+l+1)} + \frac{1}{2L+1} \right] \\ &+ \frac{1}{2\pi} \sum_{l=0}^{L-1} \frac{2l+1}{(L-l)(L+l+1)} P_l(\mathbf{n} \cdot \mathbf{n}'). \end{aligned} \quad (4.1)$$

In the particular case $L=0$, when the spherical Helmholtz operator (1.4) reduces to the spherical Laplacian, Eq. (4.1) yields

$$\bar{G}_0(\mathbf{n}, \mathbf{n}') = \frac{1}{4\pi} \left(\ln \frac{1 - \mathbf{n} \cdot \mathbf{n}'}{2} + 1 \right). \quad (4.2)$$

This result agrees with that of Kneser¹⁹ (see also the paper by Freeden¹⁴), after rescaling his formula to achieve the consistency with our defining Eq. (1.13), but at the first sight seems to contradict the finding of Courant and Hilbert,¹² whose result, again after due rescaling, in our notation is

$$\bar{G}_0^{(\text{CH})}(\mathbf{n}, \mathbf{n}') = \frac{1}{4\pi} \left(\ln \frac{1 - \mathbf{n} \cdot \mathbf{n}'}{2} + 2 \ln 2 \right). \quad (4.3)$$

The reason for this discrepancy is that the latter authors did not impose the orthogonality constraint (1.14) on the solution to the inhomogeneous equation (1.13), specialized to the case $L=0$. Consequently, in their approach the generalized Green's function for the spherical Laplace operator is determined only up to a multiple of $Y_{00}(\mathbf{n})Y_{00}^*(\mathbf{n}')$, i.e., up to an additive constant. If this constant is added to $\bar{G}_0^{(\text{CH})}(\mathbf{n}, \mathbf{n}')$, and then chosen so that the result satisfies the constraint (1.14), our finding (4.2) is recovered.

Finally, we find it noteworthy that Eqs. (2.7), (3.11), and (3.14) imply that the function $\bar{G}_L(\mathbf{n}, \mathbf{n}')$ satisfies the three-term inhomogeneous recurrence relation

$$\begin{aligned} & (L+1)\bar{G}_{L+1}(\mathbf{n}, \mathbf{n}') - (2L+1)\mathbf{n} \cdot \mathbf{n}' \bar{G}_L(\mathbf{n}, \mathbf{n}') + L\bar{G}_{L-1}(\mathbf{n}, \mathbf{n}') \\ &= \frac{1}{4\pi} \left[\frac{P_{L+1}(\mathbf{n} \cdot \mathbf{n}')}{(2L+1)(2L+3)} + \frac{P_{L-1}(\mathbf{n} \cdot \mathbf{n}')}{|2L-1|(2L+1)} \right], \end{aligned} \quad (4.4)$$

subject to the initial condition constituted by Eq. (4.2).

APPENDIX: CLOSED FORM OF THE GENERALIZED GREEN'S FUNCTION FOR THE LEGENDRE OPERATOR (1.15)

The Legendre Green's function $g(\lambda; x, x')$ is defined as the solution to the inhomogeneous differential equation (with x' fixed)

$$\left[\frac{d}{dx} (1-x^2) \frac{d}{dx} + \lambda(\lambda+1) \right] g(\lambda; x, x') = \delta(x-x') \quad (-1 < x, x' < 1) \quad (A1)$$

satisfying the boundary conditions

$$g(\lambda; x, x') \text{ bounded for } x \rightarrow \pm 1. \quad (A2)$$

In Eq. (A1), and hereafter, $\delta(x-x')$ is the one-dimensional Dirac delta distribution. The function $g(\lambda; x, x')$ has the spectral series representation

$$g(\lambda; x, x') = \frac{1}{2} \sum_{l=0}^{\infty} \frac{2l+1}{\lambda(\lambda+1) - l(l+1)} P_l(x) P_l(x') \quad (\lambda \notin \mathbb{Z}), \quad (A3)$$

while its closed form may be shown to be

$$g(\lambda; x, x') = \frac{\pi}{2 \sin(\pi\lambda)} P_\lambda(-x_<) P_\lambda(x_>) \quad (\lambda \notin \mathbb{Z}), \quad (A4)$$

with

$$x_< = \min(x, x'), \quad x_> = \max(x, x'). \quad (A5)$$

Evidently, $g(\lambda; x, x')$ fails to exist if $\lambda \in \mathbb{Z}$, i.e., if the condition (1.9) is satisfied. In this appendix, with the help of the results of Sec. III, we shall find the closed form of the generalized Legendre Green's function $\bar{g}_L(x, x')$ for this case.

The function $\bar{g}_L(x, x')$ is defined as this particular solution to the inhomogeneous equation (with x' fixed)

$$\left[\frac{d}{dx}(1-x^2) \frac{d}{dx} + L(L+1) \right] \bar{g}_L(x, x') = \delta(x-x') - \frac{2L+1}{2} P_L(x) P_L(x') \quad (-1 < x, x' < 1), \quad (\text{A6})$$

which satisfies the boundary conditions

$$\bar{g}_L(x, x') \text{ bounded for } x \rightarrow \pm 1 \quad (\text{A7})$$

and the orthogonality constraint

$$\int_{-1}^1 dx P_L(x) \bar{g}_L(x, x') = 0. \quad (\text{A8})$$

The spectral series representation of $\bar{g}_L(x, x')$ is

$$\bar{g}_L(x, x') = \frac{1}{2} \sum_{\substack{l=0 \\ (l \neq L)}}^{\infty} \frac{2l+1}{L(L+1) - l(l+1)} P_l(x) P_l(x') \quad (\text{A9})$$

and this particular form of $\bar{g}_L(x, x')$ has been helpful in deriving Eq. (3.32).

To find the closed form of $\bar{g}_L(x, x')$, we observe that, as it follows from Eqs. (A3) and (A9), it is related to the Green's function $g(\lambda; x, x')$ through

$$\bar{g}_L(x, x') = \lim_{\lambda(\lambda+1) \rightarrow L(L+1)} \frac{\partial}{\partial [\lambda(\lambda+1)]} \{ [\lambda(\lambda+1) - L(L+1)] g(\lambda; x, x') \}. \quad (\text{A10})$$

In virtue of Eq. (A4), this relation may be rewritten as

$$\bar{g}_L(x, x') = \frac{\pi}{2} \frac{1}{2L+1} \lim_{\lambda \rightarrow L} \frac{\partial}{\partial \lambda} \frac{(\lambda-L)(\lambda+L+1)}{\sin(\pi\lambda)} P_\lambda(-x_{<}) P_\lambda(x_{>}). \quad (\text{A11})$$

Performing the limiting passage with the aid of the l'Hospital rule gives

$$\bar{g}_L(x, x') = \frac{(-)^L}{2} \frac{\partial P_\lambda(-x_{<})}{\partial \lambda} \Big|_{\lambda=L} P_L(x_{>}) + \frac{(-)^L}{2} P_L(-x_{<}) \frac{\partial P_\lambda(x_{>})}{\partial \lambda} \Big|_{\lambda=L} + \frac{1}{2} \frac{P_L(x_{<}) P_L(x_{>})}{2L+1}, \quad (\text{A12})$$

and transforming further Eq. (A12) with the help of Eqs. (3.15), (3.21), and (2.6) leads to the final result

$$\begin{aligned} \bar{g}_L(x, x') &= \frac{1}{2} P_L(x_{<}) P_L(x_{>}) \ln \frac{(1-x_{<})(1+x_{>})}{4} + \frac{1}{2} W_L(-x_{<}) P_L(-x_{>}) + \frac{1}{2} P_L(x_{<}) W_L(x_{>}) \\ &+ \frac{1}{2} \frac{P_L(x_{<}) P_L(x_{>})}{2L+1}, \end{aligned} \quad (\text{A13})$$

with $W_L(x)$ given explicitly by Eq. (3.35). In the particular case $L=0$, Eq. (A13) simplifies to the well-known formula^{12,20}

$$\bar{g}_0(x, x') = \frac{1}{2} \ln \frac{(1-x_{<})(1+x_{>})}{4} + \frac{1}{2}. \quad (\text{A14})$$

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