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# Some summation formulae for spherical spinors

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**Abstract**
Two families of series with terms involving spherical spinors (spinor spherical harmonics) for a spin one-half particle,

$$\Gamma_{\pm}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \frac{\Omega_{\pm\varkappa\mu}(\boldsymbol{n})\Omega^{\dagger}_{\varkappa\mu}(\boldsymbol{n}')}{\varkappa-\kappa} \qquad (\kappa \neq \pm 1, \pm 2, \ldots)$$

and

$$\bar{\Gamma}_{\pm}(k; \boldsymbol{n}, \boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0,k)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \frac{\Omega_{\pm\varkappa\mu}(\boldsymbol{n})\Omega^{\dagger}_{\varkappa\mu}(\boldsymbol{n}')}{\varkappa-k} \qquad (k = \pm 1, \pm 2, \ldots),$$

are summed to closed forms involving the Legendre function of the first kind and its derivatives with respect to an argument and/or an index. In particular, it is shown that the closed forms of the series $\Gamma_{\pm}(0; \boldsymbol{n}, \boldsymbol{n}')$ are remarkably simple:

$$\Gamma_{+}(0; \boldsymbol{n}, \boldsymbol{n}') = -\frac{1}{4\pi} I + \frac{\mathrm{i}}{4\pi} \frac{\boldsymbol{n} \times \boldsymbol{n}'}{1 - \boldsymbol{n} \cdot \boldsymbol{n}'} \cdot \boldsymbol{\sigma}$$

and

$$\Gamma_{-}(0; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\pi} \frac{\boldsymbol{n} - \boldsymbol{n}'}{1 - \boldsymbol{n} \cdot \boldsymbol{n}'} \cdot \boldsymbol{\sigma}$$

with $I$ denoting the $2 \times 2$ unit matrix and $\boldsymbol{\sigma}$ standing for the vector of Pauli matrices. Integral eigenvalue equations solved by the spherical spinors, with either $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ or $\bar{\Gamma}_{+}(k; \boldsymbol{n}, \boldsymbol{n}')$ as kernels, are provided. It is pointed out that $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ is the Green function of the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (\kappa + 1)I$, $(\kappa \neq \pm 1, \pm 2, \ldots)$, while $\bar{\Gamma}_{+}(k; \boldsymbol{n}, \boldsymbol{n}')$ is the generalized (reduced) Green function of the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (k + 1)I$, $(k = \pm 1, \pm 2, \ldots)$, with $\boldsymbol{\Lambda} = -\mathrm{i}\boldsymbol{r} \times \boldsymbol{\nabla}$.

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## 1. Introduction

Spherical spinors (sometimes also called 'spinor spherical harmonics') for spin one-half particles, denoted hereafter as $\{\Omega_{\varkappa\mu}(\boldsymbol{n})\}$, emerge in relativistic quantum mechanics in the context of the separation of the Dirac equation in spherical polar coordinates. They play here the role analogous to that which in the theory of spinless particles is played by the scalar spherical harmonics $\{Y_{lm}(\boldsymbol{n})\}$. Still, papers devoted to systematic investigations of the properties of the spherical spinors have been only occasional. As a result, much less is known about these objects than about the spherical harmonics (cf, e.g., [1–5]). With an intention to make a small step towards changing this unsatisfactory situation, in the present work we contribute to the theory of series with terms involving the spherical spinors. Specifically, we consider two families of series,

$$\Gamma_{\pm}(\kappa;\boldsymbol{n},\boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \frac{\Omega_{\pm\varkappa\mu}(\boldsymbol{n})\Omega_{\varkappa\mu}^{\dagger}(\boldsymbol{n}')}{\varkappa-\kappa} \qquad (\kappa \neq \pm 1, \pm 2, \ldots) \tag{1.1}$$

and

$$\bar{\Gamma}_{\pm}(k;\boldsymbol{n},\boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0,k)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \frac{\Omega_{\pm\varkappa\mu}(\boldsymbol{n})\Omega_{\varkappa\mu}^{\dagger}(\boldsymbol{n}')}{\varkappa-k} \qquad (k = \pm 1, \pm 2, \ldots) \tag{1.2}$$

(the dagger denotes the Hermitian conjugation), and show that they may be summed to closed forms involving the Legendre function of the first kind and derivatives of the latter with respect to an argument and/or an index. The importance of the series $\Gamma_{+}(\kappa;\boldsymbol{n},\boldsymbol{n}')$ and $\bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}')$ lies in that, as we shall show below, they are, respectively, a Green function and a generalized Green function of some partial differential operators on the unit sphere. In addition, these two series are kernels in the integral eigenvalue equations

$$\Omega_{\varkappa\mu}(\boldsymbol{n}) = (\varkappa-\kappa)\oint_{4\pi} \mathrm{d}^2\boldsymbol{n}'\, \Gamma_{+}(\kappa;\boldsymbol{n},\boldsymbol{n}')\Omega_{\varkappa\mu}(\boldsymbol{n}') \qquad (\kappa \neq \pm 1, \pm 2, \ldots) \tag{1.3}$$

and

$$\Omega_{\varkappa\mu}(\boldsymbol{n}) = (\varkappa-k)\oint_{4\pi} \mathrm{d}^2\boldsymbol{n}'\, \bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}')\Omega_{\varkappa\mu}(\boldsymbol{n}') \qquad (\varkappa \neq k = \pm 1, \pm 2, \ldots) \tag{1.4}$$

obeyed by the spherical spinors.

Definitions and these properties of the spherical spinors, the spherical harmonics and the Legendre functions of the first kind (as well as of the Legendre polynomials), which are of relevance for the present work, are given in appendices A, B and C, respectively. The constraints imposed on $\kappa$ and $k$ in equations (1.1) and (1.2), respectively, will be assumed to hold throughout the paper.

## 2. Preliminaries

### 2.1. Definitions

We shall use the symbols $\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z$ to denote the Cartesian versors, while the symbols $\boldsymbol{n}_0$ and $\boldsymbol{n}_{\pm 1}$ will denote the spherical versors, related to the former through

$$\boldsymbol{n}_0 = \boldsymbol{n}_z \qquad \boldsymbol{n}_{\pm 1} = \mp\tfrac{1}{\sqrt{2}}(\boldsymbol{n}_x \pm \mathrm{i}\boldsymbol{n}_y). \tag{2.1}$$

The symbol $\boldsymbol{n}$ will stand for the real unit vector pointing in the direction characterized in the spherical coordinate system (with its polar and azimuthal axes directed along $\boldsymbol{n}_z$ and

$\boldsymbol{n}_x$, respectively) by the polar angle $\theta$ (with $0 \leqslant \theta \leqslant \pi$) and the azimuthal angle $\varphi$ (with $0 \leqslant \varphi < 2\pi$); analogously, the real unit vector $\boldsymbol{n}'$ will be characterized by the angles $\theta'$ and $\varphi'$.

We shall use the symbol $\boldsymbol{\sigma}$ to mark the Pauli vector

$$\boldsymbol{\sigma} = \sigma_x \boldsymbol{n}_x + \sigma_y \boldsymbol{n}_y + \sigma_z \boldsymbol{n}_z \tag{2.2}$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

The symbol $I$ will denote the $2 \times 2$ unit matrix. A useful relation is

$$(\boldsymbol{n} \cdot \boldsymbol{\sigma})(\boldsymbol{n}' \cdot \boldsymbol{\sigma}) = \boldsymbol{n} \cdot \boldsymbol{n}' I + \mathrm{i}(\boldsymbol{n} \times \boldsymbol{n}') \cdot \boldsymbol{\sigma}. \tag{2.4}$$

The dimensionless orbital angular momentum operator is defined as

$$\boldsymbol{\Lambda} = -\mathrm{i}\boldsymbol{r} \times \boldsymbol{\nabla}, \tag{2.5}$$

where $\boldsymbol{r} = r\boldsymbol{n}$ and $\boldsymbol{\nabla} = \partial/\partial \boldsymbol{r}$. The relationship

$$\boldsymbol{\Lambda} f(\boldsymbol{n} \cdot \boldsymbol{n}') = -\mathrm{i} f'(\boldsymbol{n} \cdot \boldsymbol{n}')(\boldsymbol{n} \times \boldsymbol{n}') \tag{2.6}$$

holds for an arbitrary differentiable scalar function $f(\xi)$. (Throughout the paper, the prime at a scalar function denotes its first derivative with respect to an argument.)

### 2.2. *Some properties of the series (1.1) and (1.2)*

The series (1.1) and (1.2) possess some properties which we shall find useful in the considerations carried out in section 4.

Consider at first the series in equation (1.1). Premultiplying both sides of this equation with $-\boldsymbol{n} \cdot \boldsymbol{\sigma}$ and exploiting relation (A.8), one has

$$\Gamma_{\mp}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = -\boldsymbol{n} \cdot \boldsymbol{\sigma} \Gamma_{\pm}(\kappa; \boldsymbol{n}, \boldsymbol{n}'). \tag{2.7}$$

In the similar way, one finds that

$$\Gamma_{\pm}(-\kappa; \boldsymbol{n}, \boldsymbol{n}') = \Gamma_{\mp}(\kappa; \boldsymbol{n}, \boldsymbol{n}')\boldsymbol{n}' \cdot \boldsymbol{\sigma}. \tag{2.8}$$

On combining these two results, one arrives at the reflection property

$$\Gamma_{\pm}(-\kappa; \boldsymbol{n}, \boldsymbol{n}') = -\boldsymbol{n} \cdot \boldsymbol{\sigma} \Gamma_{\pm}(\kappa; \boldsymbol{n}, \boldsymbol{n}')\boldsymbol{n}' \cdot \boldsymbol{\sigma}. \tag{2.9}$$

With no difficulty, one may show that the relations analogous to those in equations (2.7)–(2.9) hold for the series (1.2):

$$\bar{\Gamma}_{\mp}(k; \boldsymbol{n}, \boldsymbol{n}') = -\boldsymbol{n} \cdot \boldsymbol{\sigma} \bar{\Gamma}_{\pm}(k; \boldsymbol{n}, \boldsymbol{n}') \tag{2.10}$$

$$\bar{\Gamma}_{\pm}(-k; \boldsymbol{n}, \boldsymbol{n}') = \bar{\Gamma}_{\mp}(k; \boldsymbol{n}, \boldsymbol{n}')\boldsymbol{n}' \cdot \boldsymbol{\sigma} \tag{2.11}$$

$$\bar{\Gamma}_{\pm}(-k; \boldsymbol{n}, \boldsymbol{n}') = -\boldsymbol{n} \cdot \boldsymbol{\sigma} \bar{\Gamma}_{\pm}(k; \boldsymbol{n}, \boldsymbol{n}')\boldsymbol{n}' \cdot \boldsymbol{\sigma}. \tag{2.12}$$

### 2.3. *The series $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ and $\bar{\Gamma}_{+}(k; \boldsymbol{n}, \boldsymbol{n}')$ as Green functions*

Finally, we shall validate the statement, made in the introduction, that the functions $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ and $\bar{\Gamma}_{+}(k; \boldsymbol{n}, \boldsymbol{n}')$ are, respectively, a Green function and a generalized Green function of some partial differential operators.

Consider the series representing the function $\Gamma_+(\kappa; \boldsymbol{n}, \boldsymbol{n}')$. Acting on it with the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (\kappa + 1)I$, after exploiting the eigenvalue equation (A.7), we have

$$[-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (\kappa + 1)I]\Gamma_+(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \Omega_{\varkappa\mu}(\boldsymbol{n})\Omega_{\varkappa\mu}^{\dagger}(\boldsymbol{n}'). \tag{2.13}$$

The series on the right-hand side of the above equation is just this one which appears in the closure relation (A.6), so that equation (2.13) may be rewritten as

$$[-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (\kappa + 1)I]\Gamma_+(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \delta^{(2)}(\boldsymbol{n} - \boldsymbol{n}')I. \tag{2.14}$$

From this, we infer that $\Gamma_+(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ is the Green function of the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (\kappa + 1)I$ (with the constraints of differentiability, finiteness and single-valuedness imposed on functions from its domain).

Next, we turn to the series representing the function $\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}')$. Proceeding as above in the case of $\Gamma_+(\kappa; \boldsymbol{n}, \boldsymbol{n}')$, we find that $\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}')$ obeys the inhomogeneous differential equation

$$[-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (k + 1)I]\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}') = \delta^{(2)}(\boldsymbol{n} - \boldsymbol{n}')I - \sum_{\mu=-|k|+1/2}^{|k|-1/2} \Omega_{k\mu}(\boldsymbol{n})\Omega_{k\mu}^{\dagger}(\boldsymbol{n}'). \tag{2.15}$$

Moreover, it is evident from equations (1.2) and (A.3) that $\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}')$ is orthogonal to the null space of the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (k + 1)I$:

$$\oint_{4\pi} \mathrm{d}^2\boldsymbol{n}\, \Omega_{k\mu}^{\dagger}(\boldsymbol{n})\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}') = 0 \qquad \left(\mu = -|k| + \frac{1}{2}, \ldots, |k| - \frac{1}{2}\right). \tag{2.16}$$

Consequently, one recognizes in $\bar{\Gamma}_+(k; \boldsymbol{n}, \boldsymbol{n}')$ the generalized (or reduced) Green function of the operator $-\boldsymbol{\sigma} \cdot \boldsymbol{\Lambda} - (k+1)I$ (again, with the differentiability, finiteness and single-valuedness conditions imposed on functions from its domain).

## 3. Auxiliary finite sums involving spherical spinors

Before proceeding to summing the series (1.1) and (1.2), in this section we shall consider two auxiliary finite sums:

$$\Sigma_{\varkappa\pm}(\boldsymbol{n}, \boldsymbol{n}') = \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \Omega_{\pm\varkappa\mu}(\boldsymbol{n})\Omega_{\varkappa\mu}^{\dagger}(\boldsymbol{n}'). \tag{3.1}$$

A method of finding closed expressions for these sums was presented by Bechler [6] (see also [7]). Here, we shall discuss an approach somewhat different from that in [6]; as a by-product, our considerations will yield an intermediate formula which we shall find useful in section 4.

We begin with observing that, because of the property (A.8), the sums $\Sigma_{\varkappa+}(\boldsymbol{n}, \boldsymbol{n}')$ and $\Sigma_{\varkappa-}(\boldsymbol{n}, \boldsymbol{n}')$ are related through

$$\Sigma_{\varkappa-}(\boldsymbol{n}, \boldsymbol{n}') = -\boldsymbol{n} \cdot \boldsymbol{\sigma}\Sigma_{\varkappa+}(\boldsymbol{n}, \boldsymbol{n}'). \tag{3.2}$$

Therefore, in what follows we shall focus primarily on the sum $\Sigma_{\varkappa+}(\boldsymbol{n}, \boldsymbol{n}')$.

It follows from equations (3.1) and (A.1) that $\Sigma_{\varkappa+}(\boldsymbol{n}, \boldsymbol{n}')$ has the following matrix structure,

$$\Sigma_{\varkappa+}(\boldsymbol{n}, \boldsymbol{n}') = \begin{pmatrix} K_{\varkappa+}^{(-)}(\boldsymbol{n}, \boldsymbol{n}') & L_{\varkappa+}^{(-)}(\boldsymbol{n}, \boldsymbol{n}') \\ L_{\varkappa+}^{(+)}(\boldsymbol{n}, \boldsymbol{n}') & K_{\varkappa+}^{(+)}(\boldsymbol{n}, \boldsymbol{n}') \end{pmatrix} \tag{3.3}$$

where

$$K_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{2\varkappa+1}\sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2}\left(\varkappa+\frac{1}{2}\pm\mu\right)Y_{l,\mu\pm1/2}(\boldsymbol{n})Y^{*}_{l,\mu\pm1/2}(\boldsymbol{n}')$$
$$= \frac{1}{2\varkappa+1}\sum_{m=-l}^{l}(\varkappa\pm m)Y_{lm}(\boldsymbol{n})Y^{*}_{lm}(\boldsymbol{n}') \tag{3.4}$$

and

$$L_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = -\frac{1}{2\varkappa+1}\sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2}\sqrt{\left(\varkappa+\frac{1}{2}\right)^2-\mu^2}\,Y_{l,\mu\pm1/2}(\boldsymbol{n})Y^{*}_{l,\mu\mp1/2}(\boldsymbol{n}')$$
$$= -\frac{1}{2\varkappa+1}\sum_{m=-l}^{l}\sqrt{(l\mp m)(l\pm m+1)}\,Y_{l,m\pm1}(\boldsymbol{n})Y^{*}_{lm}(\boldsymbol{n}'). \tag{3.5}$$

It should be noted that

$$L_{\varkappa+}^{(-)}(\boldsymbol{n},\boldsymbol{n}') = L_{\varkappa+}^{(+)*}(\boldsymbol{n}',\boldsymbol{n}). \tag{3.6}$$

Exploiting the properties (B.4) and (B.5) of the spherical harmonics, we rewrite equations (3.4) and (3.5) in the forms

$$K_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = \frac{\varkappa\pm\boldsymbol{n}_0\cdot\boldsymbol{\Lambda}}{2\varkappa+1}\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{n})Y^{*}_{lm}(\boldsymbol{n}') \tag{3.7}$$

$$L_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = \pm\frac{\sqrt{2}}{2\varkappa+1}\boldsymbol{n}_{\pm1}\cdot\boldsymbol{\Lambda}\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{n})Y^{*}_{lm}(\boldsymbol{n}') \tag{3.8}$$

which, after employing the summation formula (B.6), reduce to

$$K_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = \frac{|\varkappa|\pm\operatorname{sgn}(\varkappa)\boldsymbol{n}_0\cdot\boldsymbol{\Lambda}}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{3.9}$$

$$L_{\varkappa+}^{(\pm)}(\boldsymbol{n},\boldsymbol{n}') = \pm\frac{\operatorname{sgn}(\varkappa)}{4\pi}\sqrt{2}\boldsymbol{n}_{\pm1}\cdot\boldsymbol{\Lambda}P_l(\boldsymbol{n}\cdot\boldsymbol{n}'), \tag{3.10}$$

where $P_l(\xi)$ is the Legendre polynomial. On inserting the results (3.9) and (3.10) into the right-hand side of equation (3.3) and making use of the definitions (2.1)–(2.3), one arrives at

$$\Sigma_{\varkappa+}(\boldsymbol{n},\boldsymbol{n}') = \frac{|\varkappa|I-\operatorname{sgn}(\varkappa)\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{3.11}$$

(this particular intermediate form of the sum $\Sigma_{\varkappa+}(\boldsymbol{n},\boldsymbol{n}')$ will play a crucial role in section 4), which after making use of the differential relationship (2.6) becomes

$$\Sigma_{\varkappa+}(\boldsymbol{n},\boldsymbol{n}') = \frac{|\varkappa|}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}')I + \mathrm{i}\frac{\operatorname{sgn}(\varkappa)}{4\pi}P'_l(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{\sigma}. \tag{3.12}$$

We have already mentioned that the same representation of $\Sigma_{\varkappa+}(\boldsymbol{n},\boldsymbol{n}')$ was derived, in a somewhat different way, in [6]. An alternative form of the sum (3.1)

$$\Sigma_{\varkappa+}(\boldsymbol{n},\boldsymbol{n}') = -\frac{\operatorname{sgn}(\varkappa)}{4\pi}P'_{l-\operatorname{sgn}(\varkappa)}(\boldsymbol{n}\cdot\boldsymbol{n}')I + \frac{\operatorname{sgn}(\varkappa)}{4\pi}P'_l(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{n}'\cdot\boldsymbol{\sigma}) \tag{3.13}$$

given in [7] may be obtained from equation (3.12) by transforming the latter with the aid of the matrix identity (2.4) and the recurrence relations (C.5) and (C.6) (both with $\lambda = l$).

The closed form of the sum $\Sigma_{\varkappa-}(\boldsymbol{n}, \boldsymbol{n}')$, following from equations (3.2) and (3.13), is

$$\Sigma_{\varkappa-}(\boldsymbol{n}, \boldsymbol{n}') = \frac{\operatorname{sgn}(\varkappa)}{4\pi}[P'_{l-\operatorname{sgn}(\varkappa)}(\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n} - P'_l(\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n}']\cdot\boldsymbol{\sigma}. \tag{3.14}$$

## 4. Summation of the series (1.1) and (1.2)

Having made the above preparatory steps, in this section we shall concentrate on the main problem of this work, i.e., on summing the series (1.1) and (1.2).

### *4.1. Summation of the series (1.1)*

Consider the series (1.1). On applying equation (3.1), they may be rewritten as

$$\Gamma_{\pm}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \frac{\Sigma_{\varkappa\pm}(\boldsymbol{n}, \boldsymbol{n}')}{\varkappa - \kappa} \qquad (\kappa \neq \pm 1, \pm 2, \ldots). \tag{4.1}$$

Since $\Gamma_{-}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ is related to $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$ via equation (2.7), below we shall present in detail the procedure of summing $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$; for $\Gamma_{-}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$, we shall provide only the final result.

To sum the series $\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}')$, we observe that, after making use of the result (3.11), equation (4.1), specified to this particular case, may be rewritten in the form

$$\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\pi} \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \frac{|\varkappa| I - \operatorname{sgn}(\varkappa)\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}}{\varkappa - \kappa} P_l(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{4.2}$$

with $l$ defined in equation (A.2). Changing the summation index from $\varkappa$ to $l$ and employing relation (2.6) transforms equation (4.2) into

$$\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\pi} A_{+}(\kappa; \boldsymbol{n}\cdot\boldsymbol{n}') I + \frac{\mathrm{i}}{4\pi} B'_{+}(\kappa; \boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{\sigma} \tag{4.3}$$

with

$$A_{+}(\kappa; \xi) = \kappa \sum_{l=0}^{\infty} \frac{2l+1}{(l-\kappa)(l+\kappa+1)} P_l(\xi) \tag{4.4}$$

$$B'_{+}(\kappa; \xi) = \sum_{l=1}^{\infty} \frac{2l+1}{(l-\kappa)(l+\kappa+1)} P'_l(\xi). \tag{4.5}$$

Applying the summation formula (C.23) to the right-hand sides of equations (4.4) and (4.5) yields

$$A_{+}(\kappa; \xi) = -\frac{\pi\kappa}{\sin(\pi\kappa)} P_{\kappa}(-\xi) \tag{4.6}$$

$$B'_{+}(\kappa; \xi) = \frac{\pi}{\sin(\pi\kappa)} P'_{\kappa}(-\xi) \tag{4.7}$$

where $P_{\kappa}(\xi)$ is the Legendre function of the first kind, defined in equation (C.1). Substituting these two results into equation (4.3), we find

$$\Gamma_{+}(\kappa; \boldsymbol{n}, \boldsymbol{n}') = -\frac{\kappa}{4\sin(\pi\kappa)} P_{\kappa}(-\boldsymbol{n}\cdot\boldsymbol{n}') I + \frac{\mathrm{i}}{4\sin(\pi\kappa)} P'_{\kappa}(-\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{\sigma} \qquad (\kappa \neq \pm 1, \pm 2, \ldots). \tag{4.8}$$

It is worthwhile observing that from equation (4.8) and property (C.3) one may deduce that

$$\Gamma_+(-\kappa-1;\boldsymbol{n},\boldsymbol{n}') = \frac{2\kappa+1}{4\sin(\pi\kappa)} P_\kappa(-\boldsymbol{n}\cdot\boldsymbol{n}')I + \Gamma_+(\kappa;\boldsymbol{n},\boldsymbol{n}'). \tag{4.9}$$

The case $\kappa = 0$ requires a more thorough investigation. One has

$$A_+(0;\xi) = -1 \tag{4.10}$$

and

$$B'_+(0;\xi) = \lim_{\kappa\to 0} \frac{P'_\kappa(-\xi)}{\kappa}. \tag{4.11}$$

However, it is evident that

$$\lim_{\kappa\to 0} \frac{P'_\kappa(-\xi)}{\kappa} = \left.\frac{\partial P'_\kappa(-\xi)}{\partial\kappa}\right|_{\kappa=0} \tag{4.12}$$

and in appendix C it is shown that

$$\left.\frac{\partial P'_\kappa(\xi)}{\partial\kappa}\right|_{\kappa=0} = \frac{1}{1+\xi}. \tag{4.13}$$

Consequently, we have

$$B'_+(0;\xi) = \frac{1}{1-\xi} \tag{4.14}$$

and from equations (4.3), (4.10) and (4.14) we derive the remarkably simple result

$$\Gamma_+(0;\boldsymbol{n},\boldsymbol{n}') = -\frac{1}{4\pi}I + \frac{\mathrm{i}}{4\pi}\frac{\boldsymbol{n}\times\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\cdot\boldsymbol{\sigma}. \tag{4.15}$$

From this it follows that for $\kappa = 0$ the integral eigenvalue equation (1.3) becomes

$$(1-\delta_{\varkappa,-1})\Omega_{\varkappa\mu}(\boldsymbol{n}) = \mathrm{i}\frac{\varkappa}{4\pi}\oint_{4\pi}\mathrm{d}^2\boldsymbol{n}'\,\frac{\boldsymbol{n}\times\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\cdot\boldsymbol{\sigma}\Omega_{\varkappa\mu}(\boldsymbol{n}'). \tag{4.16}$$

(Parenthetically, we note that one might arrive at equation (4.14) following a route alternative to that presented above. Indeed, from equations (2.9), (4.3) and (2.4), one infers that

$$A_+(-\kappa;\xi) = -\xi A_+(\kappa;\xi) - (1-\xi^2)B'_+(\kappa;\xi) \tag{4.17}$$

$$B'_+(-\kappa;\xi) = -A_+(\kappa;\xi) + \xi B'_+(\kappa;\xi). \tag{4.18}$$

In the particular case $\kappa = 0$, either of equations (4.17) or (4.18) yields

$$B'_+(0;\xi) = -\frac{A_+(0;\xi)}{1-\xi}. \tag{4.19}$$

Hence, after employing relation (4.10), equation (4.14) follows again.)

The closed form of the sum $\Gamma_-(\kappa;\boldsymbol{n},\boldsymbol{n}')$ may be obtained on combining equations (2.7) and (4.8) and then simplifying the result with the aid of the property (C.5); this yields

$$\begin{aligned}\Gamma_-(\kappa;\boldsymbol{n},\boldsymbol{n}') &= -\frac{1}{4\sin(\pi\kappa)}[P'_{\kappa-1}(-\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n} + P'_\kappa(-\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n}']\cdot\boldsymbol{\sigma}\\ &= -\frac{1}{4\sin(\pi\kappa)}[P'_{-\kappa}(-\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n} + P'_\kappa(-\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n}']\cdot\boldsymbol{\sigma} \qquad (\kappa\neq\pm1,\pm2,\ldots)\end{aligned} \tag{4.20}$$

where the second equality in the above equation follows from relation (C.3). From this, one infers the general relationship

$$\Gamma_-(-\kappa;\boldsymbol{n},\boldsymbol{n}') = -\Gamma_-(\kappa;\boldsymbol{n}',\boldsymbol{n}) \tag{4.21}$$

supplementing these in equations (2.7)–(2.9). Finally, it stems from equations (2.7) and (4.15), and also from equations (4.20) and (4.13), that $\Gamma_-(0;\boldsymbol{n},\boldsymbol{n}')$ has the particularly simple form

$$\Gamma_-(0;\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}\frac{\boldsymbol{n}-\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\cdot\boldsymbol{\sigma}. \tag{4.22}$$

*4.2. Summation of the series (1.2)*

We turn to the series (1.2) which, with the aid of equation (3.1), may be rewritten in the form

$$\bar{\Gamma}_{\pm}(k;\boldsymbol{n},\boldsymbol{n}') = \sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0,k)}}^{\infty} \frac{\Sigma_{\varkappa\pm}(\boldsymbol{n},\boldsymbol{n}')}{\varkappa - k} \qquad (k = \pm 1, \pm 2, \ldots). \tag{4.23}$$

Because of the property (2.10), we shall focus here on $\bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}')$, providing for $\bar{\Gamma}_{-}(k;\boldsymbol{n},\boldsymbol{n}')$ the final result only.

To sum the series $\bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}')$, we observe that it is related to the series $\Gamma_{+}(\kappa;\boldsymbol{n},\boldsymbol{n}')$, considered in the preceding subsection, through

$$\bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}') = \lim_{\kappa\to k} \frac{\partial}{\partial\kappa}[(\kappa - k)\Gamma_{+}(\kappa;\boldsymbol{n},\boldsymbol{n}')]. \tag{4.24}$$

After combining equation (4.24) with equation (4.8) and exploiting the l'Hospital rule, we have

$$\bar{\Gamma}_{+}(k;\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}\bar{A}_{+}(k;\boldsymbol{n}\cdot\boldsymbol{n}')I + \frac{\mathrm{i}}{4\pi}\bar{B}'_{+}(k;\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{\sigma} \tag{4.25}$$

where

$$\bar{A}_{+}(k;\xi) = (-)^{k+1}P_k(-\xi) + (-)^{k+1}k\left.\frac{\partial P_\kappa(-\xi)}{\partial\kappa}\right|_{\kappa=k} \tag{4.26}$$

$$\bar{B}'_{+}(k;\xi) = (-)^{k}\left.\frac{\partial P'_\kappa(-\xi)}{\partial\kappa}\right|_{\kappa=k}. \tag{4.27}$$

With the aid of the relations (2.12) and (2.4), from equation (4.25) one may deduce that the coefficients (4.26) and (4.27) obey

$$\bar{A}_{+}(-k;\xi) = -\xi\bar{A}_{+}(k;\xi) - (1-\xi^2)\bar{B}'_{+}(k;\xi) \tag{4.28}$$

$$\bar{B}'_{+}(-k;\xi) = -\bar{A}_{+}(k;\xi) + \xi\bar{B}'_{+}(k;\xi) \tag{4.29}$$

(cf equations (4.17) and (4.18)).

The following compact closed form of $\bar{\Gamma}_{-}(k;\boldsymbol{n},\boldsymbol{n}')$

$$\bar{\Gamma}_{-}(k;\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}[\bar{B}'_{+}(-k;\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n} - \bar{B}'_{+}(k;\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n}']\cdot\boldsymbol{\sigma} \tag{4.30}$$

results from combining equations (2.10) and (4.25) with the relation (4.29). From equation (4.30), the property

$$\bar{\Gamma}_{-}(-k;\boldsymbol{n},\boldsymbol{n}') = -\bar{\Gamma}_{-}(k;\boldsymbol{n}',\boldsymbol{n}) \tag{4.31}$$

(cf equation (4.21)) follows immediately.

We provide some examples. If $k = -1$, then from equations (4.25)–(4.27), (C.11) and (C.19) one finds that $\bar{\Gamma}_{+}(-1;\boldsymbol{n},\boldsymbol{n}')$, i.e., the generalized Green function for the operator $-\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}$, is

$$\bar{\Gamma}_{+}(-1;\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}\left(\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + 1\right)I + \frac{\mathrm{i}}{4\pi}\frac{\boldsymbol{n}\times\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\cdot\boldsymbol{\sigma}. \tag{4.32}$$

The counterpart expression for $\bar{\Gamma}_{-}(-1;\boldsymbol{n},\boldsymbol{n}')$, resulting for instance from equations (2.10), (4.32) and (2.4), is

$$\bar{\Gamma}_{-}(-1;\boldsymbol{n},\boldsymbol{n}') = -\frac{1}{4\pi}\left[\left(\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + \frac{1-2\boldsymbol{n}\cdot\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\right)\boldsymbol{n} + \frac{\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\right]\cdot\boldsymbol{\sigma}. \tag{4.33}$$

If $k = +1$, then either from equations (4.25)–(4.27), (C.17) and (C.20), or, alternatively, from equations (2.12), (4.32) and (2.4), one deduces that the closed form of $\bar{\Gamma}_{+}(+1; \boldsymbol{n}, \boldsymbol{n}')$, i.e., of the generalized Green function for the operator $-\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda} - 2I$, is given by

$$\bar{\Gamma}_{+}(+1; \boldsymbol{n}, \boldsymbol{n}') = -\frac{1}{4\pi}\left(\boldsymbol{n}\cdot\boldsymbol{n}' \ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + 2\boldsymbol{n}\cdot\boldsymbol{n}' + 1\right) I - \frac{\mathrm{i}}{4\pi}\left(\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + \frac{1-2\boldsymbol{n}\cdot\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\right)(\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{\sigma}. \tag{4.34}$$

The closed representation of the related sum $\bar{\Gamma}_{-}(+1; \boldsymbol{n}, \boldsymbol{n}')$ is

$$\bar{\Gamma}_{-}(+1; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\pi}\left[\frac{\boldsymbol{n}}{1-\boldsymbol{n}\cdot\boldsymbol{n}'} + \left(\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + \frac{1-2\boldsymbol{n}\cdot\boldsymbol{n}'}{1-\boldsymbol{n}\cdot\boldsymbol{n}'}\right)\boldsymbol{n}'\right]\cdot\boldsymbol{\sigma}, \tag{4.35}$$

as may be inferred, for instance, from equations (4.31) and (4.33).

## Appendix A. Spherical spinors

Spherical spinors are defined as two-component objects, expressed in terms of the spherical harmonics (cf appendix B) in the following way:

$$\Omega_{\varkappa\mu}(\boldsymbol{n}) = \begin{pmatrix} \operatorname{sgn}(-\varkappa)\sqrt{\frac{\varkappa+\frac{1}{2}-\mu}{2\varkappa+1}}\, Y_{l,\mu-1/2}(\boldsymbol{n}) \\ \sqrt{\frac{\varkappa+\frac{1}{2}+\mu}{2\varkappa+1}}\, Y_{l,\mu+1/2}(\boldsymbol{n}) \end{pmatrix} \tag{A.1}$$

where $\varkappa = \pm 1, \pm 2, \ldots$, $\mu = -|\varkappa| + \frac{1}{2}, -|\varkappa| + \frac{3}{2}, \ldots, |\varkappa| - \frac{1}{2}$, and

$$l \equiv l(\varkappa) = \left|\varkappa + \tfrac{1}{2}\right| - \tfrac{1}{2} = \begin{cases} -\varkappa - 1 & \text{for } \varkappa < 0 \\ \varkappa & \text{for } \varkappa > 0. \end{cases} \tag{A.2}$$

The spherical spinors are orthonormal on the unit sphere in the sense of

$$\oint_{4\pi} \mathrm{d}^2\boldsymbol{n}\, \Omega^{\dagger}_{\varkappa\mu}(\boldsymbol{n})\Omega_{\varkappa'\mu'}(\boldsymbol{n}) = \delta_{\varkappa\varkappa'}\delta_{\mu\mu'}. \tag{A.3}$$

The following summation formula holds,

$$\sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \Omega_{\varkappa\mu}(\boldsymbol{n})\Omega^{\dagger}_{\varkappa\mu}(\boldsymbol{n}') = \pi^{-1}\delta(1-\boldsymbol{n}\cdot\boldsymbol{n}')I, \tag{A.4}$$

where $\delta(\xi)$ is the Dirac delta function[1]. The factor in front of the unit matrix on the right-hand side of the above equation may be interpreted as the two-dimensional Dirac delta function on the unit sphere,

$$\pi^{-1}\delta(1-\boldsymbol{n}\cdot\boldsymbol{n}') = \delta^{(2)}(\boldsymbol{n}-\boldsymbol{n}'), \tag{A.5}$$

so that one has

$$\sum_{\substack{\varkappa=-\infty \\ (\varkappa\neq 0)}}^{\infty} \sum_{\mu=-|\varkappa|+1/2}^{|\varkappa|-1/2} \Omega_{\varkappa\mu}(\boldsymbol{n})\Omega^{\dagger}_{\varkappa\mu}(\boldsymbol{n}') = \delta^{(2)}(\boldsymbol{n}-\boldsymbol{n}')I. \tag{A.6}$$

[1] To avoid misunderstandings, we emphasize that in this paper we adopt this definition of the one-dimensional Dirac delta function, according to which for any test function $f(\xi)$ it holds that

$$\int_{\xi_0}^{\xi_1} \mathrm{d}\xi\, \delta(\xi-\xi_1) f(\xi) = \frac{1}{2} f(\xi_1) \quad (\xi_0 < \xi_1).$$

Equations (A.4) and (A.6) are alternative forms of the closure (completeness) relation for the spherical spinors.

The spherical spinors $\{\Omega_{\varkappa\mu}(\boldsymbol{n})\}$ are eigenfunctions of the matrix differential operator $\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}$, associated with the $2|\varkappa|$-fold degenerate (with respect to $\mu$) eigenvalues $\{-\varkappa-1\}$:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}\Omega_{\varkappa\mu}(\boldsymbol{n}) = -(\varkappa+1)\Omega_{\varkappa\mu}(\boldsymbol{n}). \tag{A.7}$$

In several places in the main text, the following property has been used:

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}\Omega_{\varkappa\mu}(\boldsymbol{n}) = -\Omega_{-\varkappa\mu}(\boldsymbol{n}). \tag{A.8}$$

An overview of other properties of spherical spinors may be found in [4].

## Appendix B. Spherical harmonics

The spherical harmonics used throughout this paper are defined as

$$Y_{lm}(\boldsymbol{n}) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi} \tag{B.1}$$

with $l = 0, 1, 2, \ldots, m = -l, -l+1, \ldots, l$, and with $P_l^m(\xi)$ being the associated Legendre function:

$$P_l^m(\xi) = \frac{(-)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}(\xi^2-1)^l}{\mathrm{d}\xi^{l+m}} \qquad (-1 \leqslant \xi \leqslant 1). \tag{B.2}$$

Definitions (B.1) and (B.2) conform to the Condon–Shortley phase convention [8]. The harmonics (B.1) form an orthonormal set on the unit sphere:

$$\oint_{4\pi}\mathrm{d}^2\boldsymbol{n}\,Y_{lm}^*(\boldsymbol{n})Y_{l'm'}(\boldsymbol{n}) = \delta_{ll'}\delta_{mm'}. \tag{B.3}$$

They obey the differential relations

$$\boldsymbol{n}_0\cdot\boldsymbol{\Lambda}Y_{lm}(\boldsymbol{n}) = mY_{lm}(\boldsymbol{n}) \tag{B.4}$$

$$\mp\sqrt{2}\boldsymbol{n}_{\pm1}\cdot\boldsymbol{\Lambda}Y_{lm}(\boldsymbol{n}) = \sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm1}(\boldsymbol{n}) \tag{B.5}$$

and satisfy the following so-called addition theorem:

$$\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{n})Y_{lm}^*(\boldsymbol{n}') = \frac{2l+1}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{B.6}$$

The relationship

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{n})Y_{lm}^*(\boldsymbol{n}') = \delta^{(2)}(\boldsymbol{n}-\boldsymbol{n}') \tag{B.7}$$

(with $\delta^{(2)}(\boldsymbol{n}-\boldsymbol{n}')$ to be understood in the sense of equation (A.5)) expressess the completeness of the spherical harmonics in the space of single-valued square integrable functions on the unit sphere; it is the counterpart of the relation (A.6).

## Appendix C. Legendre functions of the first kind and Legendre polynomials

In this appendix, we list these well-known (and derive these less-known) properties of the Legendre functions of the first kind and the Legendre polynomials, which have been helpful in the considerations carried out in sections 3 and, especially, 4.

The Legendre function of the first kind is defined as

$$P_\lambda(\xi) = {}_2F_1\left(-\lambda, \lambda+1; 1; \frac{1-\xi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-\lambda)_n(\lambda+1)_n}{(n!)^2}\left(\frac{1-\xi}{2}\right)^n \qquad (-1 \leqslant \xi \leqslant 1) \tag{C.1}$$

where

$$(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)} \tag{C.2}$$

is the Pochhammer symbol [9]; the constraint imposed on $\xi$ in equation (C.1) will be assumed to hold throughout the rest of this appendix. The function (C.1) possesses the evident property

$$P_{-\lambda-1}(\xi) = P_\lambda(\xi) \tag{C.3}$$

and satisfies the homogeneous recurrence relation

$$(\lambda+1)P_{\lambda+1}(\xi) - (2\lambda+1)\xi P_\lambda(\xi) + \lambda P_{\lambda-1}(\xi) = 0 \tag{C.4}$$

as well as the differential relations

$$\xi P'_\lambda(\xi) - \lambda P_\lambda(\xi) = P'_{\lambda-1}(\xi) \tag{C.5}$$

$$\xi P'_\lambda(\xi) + (\lambda+1)P_\lambda(\xi) = P'_{\lambda+1}(\xi). \tag{C.6}$$

The derivative of the Legendre function with respect to its index, obtained by differentiating equation (C.1) and exploiting the definition and some properties of the digamma function [9]

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)}\frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta} \tag{C.7}$$

is found to be

$$\frac{\partial P_\lambda(\xi)}{\partial\lambda} = \sum_{n=1}^{\infty}\frac{(-\lambda)_n(\lambda+1)_n}{(n!)^2}[\psi(\lambda+1+n) - \psi(\lambda+1-n)]\left(\frac{1-\xi}{2}\right)^n. \tag{C.8}$$

It follows from equation (C.3) that

$$\left.\frac{\partial P_{\lambda'}(\xi)}{\partial\lambda'}\right|_{\lambda'=-\lambda-1} = -\left.\frac{\partial P_{\lambda'}(\xi)}{\partial\lambda'}\right|_{\lambda'=\lambda}. \tag{C.9}$$

In the particular case $\lambda = 0$, the series on the right-hand side of equation (C.8) simplifies and one has

$$\left.\frac{\partial P_\lambda(\xi)}{\partial\lambda}\right|_{\lambda=0} = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1-\xi}{2}\right)^n = \ln\frac{1+\xi}{2}. \tag{C.10}$$

Combining this with the relation (C.9) gives

$$\left.\frac{\partial P_\lambda(\xi)}{\partial\lambda}\right|_{\lambda=-1} = -\ln\frac{1+\xi}{2}. \tag{C.11}$$

Differentiating the relation (C.4) with respect to $\lambda$ yields the inhomogeneous recurrence relation for the function (C.8):

$$(\lambda+1)\frac{\partial P_{\lambda+1}(\xi)}{\partial\lambda} - (2\lambda+1)\xi\frac{\partial P_{\lambda}(\xi)}{\partial\lambda} + \lambda\frac{\partial P_{\lambda-1}(\xi)}{\partial\lambda} = -P_{\lambda+1}(\xi) + 2\xi P_{\lambda}(\xi) - P_{\lambda-1}(\xi). \tag{C.12}$$

From this and from equations (C.10) and (C.11), it may be inferred that

$$\left.\frac{\partial P_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=l} = \operatorname{sgn}\left(l+\frac{1}{2}\right) P_l(\xi)\ln\frac{1+\xi}{2} + W_l(\xi) \qquad (l = 0, \pm1, \pm2, \ldots) \tag{C.13}$$

where the polynomials $W_l(\xi)$ satisfy

$$(l+1)W_{l+1}(\xi) - (2l+1)\xi W_l(\xi) + lW_{l-1}(\xi) = -P_{l+1}(\xi) + 2\xi P_l(\xi) - P_{l-1}(\xi) \tag{C.14}$$

with the initial conditions

$$W_0(\xi) = 0 \qquad W_{-1}(\xi) = 0. \tag{C.15}$$

It holds that

$$W_{-l-1}(\xi) = -W_l(\xi). \tag{C.16}$$

In particular, from equations (C.13)–(C.16) one has

$$\left.\frac{\partial P_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=1} = -\left.\frac{\partial P_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=-2} = \xi\ln\frac{1+\xi}{2} + \xi - 1. \tag{C.17}$$

It stems from equation (C.13) that

$$\left.\frac{\partial P'_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=l} = \operatorname{sgn}\left(l+\frac{1}{2}\right) P'_l(\xi)\ln\frac{1+\xi}{2} + \operatorname{sgn}\left(l+\frac{1}{2}\right)\frac{P_l(\xi)}{1+\xi} + W'_l(\xi). \tag{C.18}$$

In particular, it holds that

$$\left.\frac{\partial P'_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=0} = -\left.\frac{\partial P'_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=-1} = \frac{1}{1+\xi} \tag{C.19}$$

$$\left.\frac{\partial P'_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=1} = -\left.\frac{\partial P'_{\lambda}(\xi)}{\partial\lambda}\right|_{\lambda=-2} = \ln\frac{1+\xi}{2} + \frac{1+2\xi}{1+\xi}. \tag{C.20}$$

For $\lambda = l = 0, \pm1, \pm2, \ldots$ the Legendre function (C.1) degenerates to the Legendre polynomial

$$P_l(\xi) = \frac{1}{2^l l!}\frac{\mathrm{d}^l(\xi^2-1)^l}{\mathrm{d}\xi^l} \qquad (l = 0, 1, 2, \ldots) \tag{C.21}$$

$$P_{-l-1}(\xi) = P_l(\xi) \qquad (l = 0, 1, 2, \ldots). \tag{C.22}$$

The following summation formula [10, 11] for the Legendre polynomials,

$$\sum_{l=0}^{\infty} \frac{(2l+1)P_l(\xi)}{(l-\lambda)(l+\lambda+1)} = -\frac{\pi}{\sin(\pi\lambda)} P_\lambda(-\xi) \qquad (-1 \leqslant \xi \leqslant 1; \lambda \neq 0, \pm 1, \pm 2, \ldots), \tag{C.23}$$

has played the crucial role in passing from equation (4.3) to equation (4.8).

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