

ADDENDUM

Addendum to ‘On the derivative of the Legendre function of the first kind with respect to its degree’

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Abstract

In our recent paper (Szmytkowski 2006 *J. Phys. A: Math. Gen.* **39** 15147; 2007 *J. Phys. A: Math. Theor.* **40** 7819 (corrigendum)), we have investigated the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree. In this addendum, we derive some further representations of $[\partial P_\nu(z)/\partial \nu]_{\nu=n}$ with $n \in \mathbb{N}$. The results are used to obtain in an elementary way two formulae for the Legendre function of the second kind of integer degree, $Q_n(z)$.

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In a recent paper [1], we have extensively investigated the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree ν , with emphasis on the particular case of $\nu = n \in \mathbb{N}$. We have shown that $[\partial P_\nu(z)/\partial \nu]_{\nu=n}$ is of the form

$$\left. \frac{\partial P_\nu(z)}{\partial \nu} \right|_{\nu=n} = P_n(z) \ln \frac{z+1}{2} + R_n(z) \quad (n \in \mathbb{N}), \quad (1)$$

where $P_n(z)$ is the Legendre polynomial and $R_n(z)$ is a polynomial in z of degree n . Some previously known formulae for $R_n(z)$, due to Bromwich and Schelkunoff, have been rederived by us and a new expression has also been given.

After the paper [1] has been published, it has come to our attention that we have not fully exploited the Jolliffe’s formula

$$\left. \frac{\partial P_\nu(z)}{\partial \nu} \right|_{\nu=n} = -P_n(z) \ln \frac{z+1}{2} + \frac{1}{2^{n-1}n!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \ln \frac{z+1}{2} \right] \quad (2)$$

discussed in section 5.1 of [1]. In this addendum, it will be shown how this formula may be used to derive two further new representations of $R_n(z)$ and to rederive, in a much simpler way, the representation (5.90). We shall also obtain two expressions for the Legendre function of the second kind, $Q_n(z)$, different from those given in [1].

We begin with the observation that from (1) and (2) it follows that

$$R_n(z) = -2P_n(z) \ln \frac{z+1}{2} + \frac{1}{2^{n-1}n!} \frac{d^n}{dz^n} \left[(z-1)^n (z+1)^n \ln \frac{z+1}{2} \right]. \quad (3)$$

Obviously, this may be rewritten as

$$R_n(z) = -2P_n(z) \ln \frac{z+1}{2} + \frac{1}{2^{n-1}n!} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}(z-1)^n}{dz^{n-k}} \frac{d^k}{dz^k} \left[(z+1)^n \ln \frac{z+1}{2} \right]. \quad (4)$$

Preparing the next step, we provide here the following differentiation formula:

$$\frac{d^k}{dz^k} [z^\alpha \ln z] = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} z^{\alpha-k} \ln z + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} [\psi(\alpha+1) - \psi(\alpha-k+1)] z^{\alpha-k} \\ (k \in \mathbb{N}; \alpha \in \mathbb{C}), \quad (5)$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)} \frac{d\Gamma(\zeta)}{d\zeta} \quad (6)$$

is the digamma function. This formula may be easily proved by mathematical induction by using basic properties of the digamma function [2, 3]. Rewriting (5) in the form

$$\frac{d^k}{dz^k} [z^\alpha \ln z] = \frac{d^k z^\alpha}{dz^k} \ln z + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} [\psi(\alpha+1) - \psi(\alpha-k+1)] z^{\alpha-k} \\ (k \in \mathbb{N}; \alpha \in \mathbb{C}) \quad (7)$$

and using (7) in (4), after straightforward manipulations we obtain

$$R_n(z) = -2P_n(z) \ln \frac{z+1}{2} + \frac{1}{2^{n-1}n!} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}(z-1)^n}{dz^{n-k}} \frac{d^k(z+1)^n}{dz^k} \ln \frac{z+1}{2} \\ + 2 \left(\frac{z+1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(n+1) - \psi(n-k+1)] \left(\frac{z-1}{z+1} \right)^k. \quad (8)$$

From the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n (z^2 - 1)^n}{dz^n} \quad (9)$$

it follows that

$$P_n(z) = \frac{1}{2^n n!} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}(z-1)^n}{dz^{n-k}} \frac{d^k(z+1)^n}{dz^k}. \quad (10)$$

Hence, it is seen that the sum of the first and the second terms on the right-hand side of (8) is zero and we have

$$R_n(z) = 2 \left(\frac{z+1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(n+1) - \psi(n-k+1)] \left(\frac{z-1}{z+1} \right)^k. \quad (11)$$

If the summation variable is changed from k to $k' = n - k$, use is made of one of the elementary properties of the binomial coefficient and the primes are omitted, (11) is cast into

$$R_n(z) = 2 \left(\frac{z-1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(n+1) - \psi(k+1)] \left(\frac{z+1}{z-1} \right)^k. \quad (12)$$

(It is to be noted that the lower summation limit in (11) may be shifted to 1 while the upper summation limit in (12) to $n - 1$.) The two representations of $R_n(z)$ given in (11) and (12) supplement formulae collected in section 5.2 in [1].

We have already mentioned that the Jolliffe’s formula (2) may also be exploited to rederive in a simple manner formula (5.90) from [1]. To show this, we note that if the binomial theorem is used to expand $(z - 1)^n$ in a sum of powers of $z + 1$ and the result is inserted into (3), this yields

$$R_n(z) = -2P_n(z) \ln \frac{z+1}{2} + 2 \sum_{k=0}^n \frac{(-1)^{n+k}}{2^k k!(n-k)!} \frac{d^n}{dz^n} \left[(z+1)^{n+k} \ln \frac{z+1}{2} \right]. \tag{13}$$

By applying formula (5), after some obvious movements the above equation becomes

$$R_n(z) = -2P_n(z) \ln \frac{z+1}{2} + 2 \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!} \left(\frac{z+1}{2}\right)^k \ln \frac{z+1}{2} + 2 \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!} [\psi(n+k+1) - \psi(k+1)] \left(\frac{z+1}{2}\right)^k. \tag{14}$$

By virtue of the Murphy’s formula

$$P_n(z) = \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!} \left(\frac{z+1}{2}\right)^k, \tag{15}$$

we see that the first and the second terms on the right-hand side of (14) cancel out. This yields

$$R_n(z) = 2 \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!} [\psi(n+k+1) - \psi(k+1)] \left(\frac{z+1}{2}\right)^k \tag{16}$$

which is identical with equation (5.90) in [1].

Having established the representations (11) and (12) of $R_n(z)$, we may obtain two expressions for the Legendre function of the second kind of integer degree, $Q_n(z)$. To show this, we recall (cf [1, sections 5.3 and 6.1]) that the Christoffel polynomial $W_{n-1}(z)$ appearing in the equation

$$Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z) \quad (z \in \mathbb{C} \setminus [-1, 1]) \tag{17}$$

is related to the polynomial $R_n(z)$ through

$$W_{n-1}(z) = -\frac{1}{2} [R_n(z) - (-1)^n R_n(-z)]. \tag{18}$$

Using $R_n(z)$ in the form (11) and employing (12) to evaluate $R_n(-z)$ gives

$$W_{n-1}(z) = \left(\frac{z+1}{2}\right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \left(\frac{z-1}{z+1}\right)^k. \tag{19}$$

If the roles played by (11) and (12) are interchanged, this results in

$$W_{n-1}(z) = \left(\frac{z-1}{2}\right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(k+1) - \psi(n-k+1)] \left(\frac{z+1}{z-1}\right)^k. \tag{20}$$

Inserting (19) and (20) into (17) leads to the following representations of $Q_n(z)$:

$$Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} \mp \left(\frac{z \pm 1}{2}\right)^n \sum_{k=0}^n \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \left(\frac{z \mp 1}{z \pm 1}\right)^k \quad (z \in \mathbb{C} \setminus [-1, 1]). \tag{21}$$

These representations of $Q_n(z)$ are seen to be direct counterparts of the following representations of $P_n(z)$ [4, p 17]:

$$P_n(z) = \left(\frac{z \pm 1}{2}\right)^n {}_2F_1\left(-n, -n; 1; \frac{z \mp 1}{z \pm 1}\right) = \left(\frac{z \pm 1}{2}\right)^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{z \mp 1}{z \pm 1}\right)^k. \quad (22)$$

For $z = x \in [-1, 1]$, the Legendre function of the second kind of integer degree may be expressed as

$$Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x) \quad (-1 \leq x \leq 1), \quad (23)$$

where

$$W_{n-1}(x) = W_{n-1}(z = x). \quad (24)$$

Adopting the common parametrization

$$x = \cos \theta \quad (0 \leq \theta \leq \pi) \quad (25)$$

from (23), (24), (19) and (20) we obtain

$$Q_n(\cos \theta) = P_n(\cos \theta) \ln \cot \frac{\theta}{2} - \cos^{2n} \frac{\theta}{2} \sum_{k=0}^n (-1)^k \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \tan^{2k} \frac{\theta}{2} \quad (26)$$

and

$$Q_n(\cos \theta) = P_n(\cos \theta) \ln \cot \frac{\theta}{2} + \sin^{2n} \frac{\theta}{2} \sum_{k=0}^n (-1)^{n+k} \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \cot^{2k} \frac{\theta}{2}. \quad (27)$$

These representations of $Q_n(\cos \theta)$ harmonize with the following known [4, p 18] expressions for $P_n(\cos \theta)$:

$$\begin{aligned} P_n(\cos \theta) &= \cos^{2n} \frac{\theta}{2} {}_2F_1\left(-n, -n; 1; -\tan^2 \frac{\theta}{2}\right) \\ &= \cos^{2n} \frac{\theta}{2} \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \tan^{2k} \frac{\theta}{2} \end{aligned} \quad (28)$$

and

$$\begin{aligned} P_n(\cos \theta) &= (-1)^n \sin^{2n} \frac{\theta}{2} {}_2F_1\left(-n, -n; 1; -\cot^2 \frac{\theta}{2}\right) \\ &= \sin^{2n} \frac{\theta}{2} \sum_{k=0}^n (-1)^{n+k} \binom{n}{k}^2 \cot^{2k} \frac{\theta}{2}, \end{aligned} \quad (29)$$

respectively.

We are not aware of any appearance of either of formulae (11), (12), (21), (26) or (27) in the literature.

References

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