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Closed forms of the Green's function and the generalized Green's function for the Helmholtz operator on the N-dimensional unit sphere

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Abstract

The Green's function for the Helmholtz differential operator $\nabla^2+\lambda(\lambda+N-1)$ on the N-dimensional (with $N\geqslant 1$) hyperspherical surface $\mathbb{S}^N$ of unit radius is investigated. Its closed form is shown to be

$$G^{(N)}(\lambda;\boldsymbol{n},\boldsymbol{n}')=\frac{\pi}{(N-1)S_N\sin(\pi\lambda)}C_\lambda^{((N-1)/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}'),$$

where S_N is the area of $\mathbb{S}^N$, $C_\lambda^{(\alpha)}(x)$ is the Gegenbauer function of the first kind, while $\boldsymbol{n}$ and $\boldsymbol{n}'$ are radius vectors, with respect to the centre of $\mathbb{S}^N$, of the observation and source points, respectively. The Green's function $G^{(N)}(\lambda;\boldsymbol{n},\boldsymbol{n}')$ fails to exist whenever λ is such that it holds that $\lambda(\lambda+N-1)=L(L+N-1)$, with $L\in\mathbb{N}$. For these exceptional cases, the generalized (known also as 'modified' or 'reduced') Green's function $\bar{G}_L^{(N)}(\boldsymbol{n},\boldsymbol{n}')$ is considered. It is shown that $\bar{G}_L^{(N)}(\boldsymbol{n},\boldsymbol{n}')$ may be expressed compactly in terms of the Gegenbauer polynomial $C_L^{((N-1)/2)}(\boldsymbol{n}\cdot\boldsymbol{n}')$ and the derivative $\left[\partial C_\lambda^{((N-1)/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}')/\partial\lambda\right]_{\lambda=L}$. Explicit expressions for the derivatives $\left[\partial C_\lambda^{(n)}(x)/\partial\lambda\right]_{\lambda=L}$ and $\left[\partial C_\lambda^{(n+1/2)}(x)/\partial\lambda\right]_{\lambda=L}$, with $n\in\mathbb{N}$, are found and used to transform the functions $\bar{G}_L^{(2n+1)}(\boldsymbol{n},\boldsymbol{n}')$ and $\bar{G}_L^{(2n+2)}(\boldsymbol{n},\boldsymbol{n}')$ to potentially more useful forms.

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1. Introduction

The Green's functions technique is known to be one of the most valuable mathematical tools used in theoretical physics [1–10]. Because of this, Green's functions, particularly those for partial differential operators of mathematical physics, are themselves frequent objects of studies. The present paper contributes to the theory of Green's functions for the Helmholtz operator. Specifically, we shall investigate the Green's function and, in those particular cases where the latter fails to exists, the generalized Green's function for the Helmholtz operator on the N-dimensional (with $N \geqslant 1$) hypersphere $\mathbb{S}^N$ in the Euclidean space $\mathbb{R}^{N+1}$ [11].

The motivation for the present study has arisen in the course of a research aimed at developing a variant of the perturbation theory for atomic processes within the framework of the adiabatic hyperspherical formalism [12–21]. The approach (to be described in detail elsewhere) starts with converting a differential eigenproblem for the adiabatic hyperspherical Hamiltonian into an integral one; this is achieved with the use of the hyperspherical Helmholtz Green's function constructed in this work.

The structure of the paper is as follows. In section 2, we recall basic facts about the hyperspherical Laplace and Helmholtz operators. Also, we summarize these properties of their eigenfunctions—hyperspherical harmonics—which shall find applications in later sections. In section 3, we arrive at the closed form of the Green's function for the hyperspherical Helmholtz operator; we also point at misprints in [22], where the analogous formula appeared. For some particular values of the propagation constant, the Green's function found in section 3 fails to exists. Then, however, one may define a generalized (known also as 'reduced' or 'modified') Green's function for the hyperspherical Helmholtz operator. We discuss this object in section 4. First, we present a compact general formula for the generalized Green's function under study; this formula contains a derivative of the Gegenbauer function of the first kind with respect to its degree. If $1 \leqslant N \leqslant 3$, the Gegenbauer function involved may be expressed in terms of the Chebyshev function of the first kind (for $N = 1$), the Legendre function of the first kind (for $N = 2$) and the Chebyshev function of the second kind (for $N = 3$), and we exploit this fact to derive more explicit forms of the generalized Helmholtz Green's functions in these particular cases. Finally, we proceed to discussing in sequence the cases of N arbitrary odd and N arbitrary even. We derive formulae for the derivatives of the Gegenbauer functions of the first kinds, $C_\lambda^{(n)}(x)$ and $C_\lambda^{(n+1/2)}(x)$, where $n \in \mathbb{N}$, with respect to their degrees, evaluated at non-negative values of the latter, i.e., for $\left[\partial C_\lambda^{(n)}(x)/\partial\lambda\right]_{\lambda=L}$ and $\left[\partial C_\lambda^{(n+1/2)}(x)/\partial\lambda\right]_{\lambda=L}$, where $L \in \mathbb{N}$. These formulae, which also seem to be of interest from the point of view of the theory of special functions, are then used to obtain explicit representations of the generalized Helmholtz Green's functions in the aforementioned two cases. The paper ends with an appendix in which we summarize those properties of the Gegenbauer, Legendre and Chebyshev functions and polynomials, which have found applications in the present work.

2. Laplace and Helmholtz operators, and their eigenfunctions, on the surface of the unit sphere in $\mathbb{R}^{N+1}$

Let $\mathbb{S}^N$, with $N \geqslant 1$, be the surface of the unit sphere in the Euclidean space $\mathbb{R}^{N+1}$. We shall denote by $\{\boldsymbol{e}_k\}$, with $k = 1, \ldots, N+1$, unit vectors of a Cartesian coordinate system in $\mathbb{R}^{N+1}$, the origin of which is located at the centre of $\mathbb{S}^N$. For any point on $\mathbb{S}^N$, its radius vector $\boldsymbol{n}$ is uniquely determined [23, 24] by specifying N real angles $\{\vartheta_k\}$ $(k = 1, \ldots, N)$ restricted by

$$0 \leqslant \vartheta_k \leqslant \pi \qquad (k = 1, 2, \ldots, N-1) \tag{2.1a}$$

$$0 \leqslant \vartheta_N < 2\pi \tag{2.1b}$$

and such that

$$\boldsymbol{e}_k \cdot \boldsymbol{n} = \cos\vartheta_k \prod_{k'=1}^{k-1} \sin\vartheta_{k'} \qquad (k = 1, 2, 3, \ldots, N) \tag{2.2a}$$

$$\boldsymbol{e}_{N+1} \cdot \boldsymbol{n} = \prod_{k'=1}^{N} \sin \vartheta_{k'}. \tag{2.2b}$$

(In equation (2.2*a*), and hereafter, we adopt the convention which states that if the upper limit of a product index is less by unity than the lower one, then the product's value equals unity.)

The infinitesimal surface element on $\mathbb{S}^N$ is

$$\mathrm{d}^N \boldsymbol{n} = \prod_{k=1}^{N} \mathrm{d}\vartheta_k \sin^{N-k} \vartheta_k \tag{2.3}$$

and the surface area of $\mathbb{S}^N$ (i.e., the total solid angle about a point in $\mathbb{R}^{N+1}$) is

$$S_N = \oint_{\mathbb{S}^N} \mathrm{d}^N \boldsymbol{n} = \frac{2\pi^{(N+1)/2}}{\Gamma\left(\frac{N+1}{2}\right)}. \tag{2.4}$$

The Laplace differential operator on $\mathbb{S}^N$ (the N-dimensional spherical Laplacian) is defined in terms of the angles $\{\vartheta_k\}$ as [23–25]

$$\begin{aligned}\nabla_{\boldsymbol{n}}^2 &= \sum_{k=1}^{N} \left(\prod_{k'=1}^{k-1} \sin^{-2} \vartheta_{k'} \right) \left(\frac{\partial^2}{\partial \vartheta_k^2} + (N-k) \cot \vartheta_k \frac{\partial}{\partial \vartheta_k} \right) \\ &= \sum_{k=1}^{N} \left(\prod_{k'=1}^{k-1} \sin^{-2} \vartheta_{k'} \right) \sin^{-(N-k)} \vartheta_k \frac{\partial}{\partial \vartheta_k} \left(\sin^{N-k} \vartheta_k \frac{\partial}{\partial \vartheta_k} \right). \end{aligned} \tag{2.5}$$

The Helmholtz differential operator on $\mathbb{S}^N$ (the N-dimensional spherical Helmholtz operator) is then defined as

$$\mathcal{H}^{(N)}(\lambda; \boldsymbol{n}) = \nabla_{\boldsymbol{n}}^2 + \lambda(\lambda + N - 1) \tag{2.6}$$

where, for the sake of later convenience, we have expressed the square of the (in general complex) propagation constant in terms of the parameter $\lambda \in \mathbb{C}$.

If the Laplacian (2.5) is constrained to operate on functions which are single-valued and non-singular on $\mathbb{S}^N$, its eigenfunctions are the hyperspherical harmonics $\left\{Y_{lm}^{(N)}(\boldsymbol{n})\right\}$ (see [24, 26–31]), which obey

$$\nabla_{\boldsymbol{n}}^2 Y_{lm}^{(N)}(\boldsymbol{n}) = -l(l + N - 1) Y_{lm}^{(N)}(\boldsymbol{n}) \qquad (l \in \mathbb{N}). \tag{2.7}$$

Evidently, it holds that

$$\left[\nabla_{\boldsymbol{n}}^2 + \lambda(\lambda + N - 1)\right] Y_{lm}^{(N)}(\boldsymbol{n}) = [\lambda(\lambda + N - 1) - l(l + N - 1)] Y_{lm}^{(N)}(\boldsymbol{n}), \tag{2.8}$$

i.e., the hyperspherical harmonics are also eigenfunctions of the Helmholtz operator (2.6). Eigenvalues in equations (2.7) and (2.8) appear to be $d_l^{(N)}$-fold degenerate, with

$$d_l^{(N)} = \begin{cases} \dfrac{(2l + N - 1)(l + N - 2)!}{l!(N-1)!} & \text{for} \quad l \geqslant 1 \\ 1 & \text{for} \quad l = 0 \end{cases} \tag{2.9}$$

and the second subscript at $Y_{lm}^{(N)}(\boldsymbol{n})$ serves to distinguish between $d_l^{(N)}$ linearly independent harmonics associated with a particular eigenvalue of $\nabla_{\boldsymbol{n}}^2$ (or $\mathcal{H}^{(N)}(\lambda; \boldsymbol{n})$).

We shall not give here the explicit expression for the hyperspherical harmonics since its knowledge is unnecessary to achieve the goal of this work; the interested reader is referred to [24, 26–31]. Instead, below we shall summarize these properties of $\left\{Y_{lm}^{(N)}(\boldsymbol{n})\right\}$ which have proved to be useful in the present context.

First, the hyperspherical harmonics may be chosen to be orthonormal over $\mathbb{S}^N$ in the sense of

$$\oint_{\mathbb{S}^N} \mathrm{d}^N \boldsymbol{n}\, Y_{lm}^{(N)*}(\boldsymbol{n}) Y_{l'm'}^{(N)}(\boldsymbol{n}) = \delta_{ll'}\delta_{mm'} \tag{2.10}$$

and throughout the rest of the work it will be assumed that the relation (2.10) holds. Second, they form a complete set in the space $L^2(\mathbb{S}^N, \mathrm{d}^N\boldsymbol{n})$ and this fact is reflected by the closure relation

$$\sum_{l=0}^{\infty} \sum_{m=1}^{d_l^{(N)}} Y_{lm}^{(N)}(\boldsymbol{n}) Y_{lm}^{(N)*}(\boldsymbol{n}') = \delta^{(N)}(\boldsymbol{n} - \boldsymbol{n}'). \tag{2.11}$$

In equation (2.11), and hereafter, $\delta^{(N)}(\boldsymbol{n} - \boldsymbol{n}')$ is the Dirac delta distribution on $\mathbb{S}^N$; it may be expressed in terms of the one-dimensional Dirac delta as[1]

$$\delta^{(N)}(\boldsymbol{n} - \boldsymbol{n}') = \frac{2}{S_{N-1}} \frac{\delta(1 - \boldsymbol{n}\cdot\boldsymbol{n}')}{[1 - (\boldsymbol{n}\cdot\boldsymbol{n}')^2]^{N/2-1}}. \tag{2.12}$$

The last property of $\{Y_{lm}^{(N)}(\boldsymbol{n})\}$ we wish to highlight here is the so-called addition theorem which states that [24, 26–28, 30]

$$\sum_{m=1}^{d_l^{(N)}} Y_{lm}^{(N)*}(\boldsymbol{n}) Y_{lm}^{(N)}(\boldsymbol{n}') = \frac{2l + N - 1}{(N-1)S_N} C_l^{((N-1)/2)}(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{2.13}$$

where $C_l^{(\alpha)}(x)$ is the Gegenbauer polynomial (cf the appendix).

3. Green's function for the Helmholtz operator on $\mathbb{S}^N$

3.1. General considerations

The Green's function, $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$, for the Helmholtz operator (2.6) is defined formally as a single-valued and finite (except for the point $\boldsymbol{n}'$ whenever $N \geqslant 2$) solution to the inhomogeneous Helmholtz differential equation ($\boldsymbol{n}'$ fixed)

$$\left[\nabla_{\boldsymbol{n}}^2 + \lambda(\lambda + N - 1)\right] G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \delta^{(N)}(\boldsymbol{n} - \boldsymbol{n}') \tag{3.1}$$

with the right-hand side being the Dirac delta distribution (2.12). Invoking facts known from the general theory of Green's functions [1–10], from equations (3.1), (2.8), (2.10) and (2.11) we deduce that $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$ has the spectral expansion

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \sum_{l=0}^{\infty} \sum_{m=1}^{d_l^{(N)}} \frac{Y_{lm}^{(N)}(\boldsymbol{n}) Y_{lm}^{(N)*}(\boldsymbol{n}')}{\lambda(\lambda + N - 1) - l(l + N - 1)}. \tag{3.2}$$

The expansion (3.2) may be simplified upon using the addition theorem (2.13); one obtains

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{(N-1)S_N} \sum_{l=0}^{\infty} \frac{2l + N - 1}{(\lambda - l)(\lambda + N + l - 1)} C_l^{((N-1)/2)}(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{3.3}$$

[1] To avoid misunderstandings, we emphasize that in this paper we adopt this particular definition of the one-dimensional Dirac delta function, according to which for any test function $f(x)$ it holds that

$$\int_{x_0}^{x_1} \mathrm{d}x\, \delta(x - x_1) f(x) = \frac{1}{2} f(x_1) \qquad (x_0 < x_1).$$

or equivalently

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{(N-1)S_N} \sum_{l=0}^{\infty} \left(\frac{1}{\lambda - l} - \frac{1}{\lambda + N + l - 1} \right) C_l^{((N-1)/2)}(\boldsymbol{n} \cdot \boldsymbol{n}'). \tag{3.4}$$

After due notational changes ($\lambda \to \alpha$, $N \to n-1$, $S_N \to \omega_{n-1}$, $\boldsymbol{n} \to \omega$, $\boldsymbol{n}' \to \upsilon$), the right-hand side of equation (3.4) should replace the misprinted expression at the bottom of p 555 in [22].

Two inferences may be drawn from equation (3.3). The first one is that the Green's function $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$ depends on the radius vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$ through their scalar product $\boldsymbol{n} \cdot \boldsymbol{n}'$ only; in other words, it holds that

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') \equiv F^{(N)}(\lambda; \cos\theta) \tag{3.5}$$

where

$$\theta = \angle(\boldsymbol{n}, \boldsymbol{n}'). \tag{3.6}$$

The second inference is that the analysis of equation (3.3) requires special care when $N = 1$; because of this, in the following subsection this particular case will be treated separately.

3.2. *The case of $N = 1$*

Separating out in equation (3.3) the term with $l = 0$, rewriting subsequently the remaining series with the formal use of the property (A.2), applying equation (A.20) and exploiting the fact that $S_1 = 2\pi$ yields $G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$ as a series of the Chebyshev polynomials of the first kind (cf the appendix):

$$G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{2\pi\lambda^2} + \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{1}{\lambda^2 - l^2} T_l(\boldsymbol{n} \cdot \boldsymbol{n}') \tag{3.7}$$

or still more explicitly

$$G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{2\pi\lambda^2} + \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{\cos[l \arccos(\boldsymbol{n} \cdot \boldsymbol{n}')]}{\lambda^2 - l^2}. \tag{3.8}$$

On the other hand, it is easily provable that the function $\cos(\lambda x)$ has the Fourier expansion

$$\cos(\lambda x) = \frac{\sin(\pi\lambda)}{\pi} \left(\frac{1}{\lambda} + 2\lambda \sum_{l=1}^{\infty} \frac{\cos[l(\pi - x)]}{\lambda^2 - l^2} \right) \qquad (-\pi \leqslant x \leqslant \pi). \tag{3.9}$$

Comparing equations (3.8) and (3.9) one finds that

$$G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{\cos[\lambda \arccos(-\boldsymbol{n} \cdot \boldsymbol{n}')]}{2\lambda \sin(\pi\lambda)} \tag{3.10}$$

or equivalently

$$G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{2\lambda \sin(\pi\lambda)} T_\lambda(-\boldsymbol{n} \cdot \boldsymbol{n}') \tag{3.11}$$

where $T_\lambda(x)$ is the Chebyshev function of the first kind (see the appendix). For the sake of later comparison with the main result of section 3.3, it is convenient to rewrite equation (3.11), with the use of relation (A.20), in the form

$$G^{(1)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{\pi}{2S_1 \sin(\pi\lambda)} C_\lambda^{(0)}(-\boldsymbol{n} \cdot \boldsymbol{n}') \tag{3.12}$$

where $C_\lambda^{(\alpha)}(x)$ is the Gegenbauer function of the first kind (cf again the appendix).

3.3. The case of $N \geqslant 2$

We proceed to discussing the case of $N \geqslant 2$. To simplify the analysis, we may exploit the property of $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$ expressed in equations (3.5) and (3.6). We choose the Cartesian basis, introduced at the beginning of section 2, in such a manner that the unit vector $\boldsymbol{e}_1$ coincides with the radius vector $\boldsymbol{n}'$ specifying the observation point. With this choice, it holds that

$$\theta = \vartheta_1 \tag{3.13}$$

so that, in virtue of equations (3.1), (2.5), (2.12) and (3.5), we have

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{N-1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + \lambda(\lambda + N - 1)\sin^{N-1}\theta\right] F^{(N)}(\lambda; \cos\theta) = \frac{2}{S_{N-1}}\sin\theta\,\delta(1 - \cos\theta) \qquad (0 \leqslant \theta \leqslant \pi). \tag{3.14}$$

The distributional equation (3.14) contains two pieces of information. First, it says that the function $F^{(N)}(\lambda; \cos\theta)$ is regular on the interval $0 < \theta \leqslant \pi$ and satisfies here the homogeneous equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{N-1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + \lambda(\lambda + N - 1)\sin^{N-1}\theta\right] F^{(N)}(\lambda; \cos\theta) = 0 \qquad (0 < \theta \leqslant \pi). \tag{3.15}$$

Second, it informs that at the end point $\theta = 0$ the following limiting relation holds:

$$\lim_{\theta \downarrow 0} \sin^{N-1}\theta \frac{\mathrm{d}F^{(N)}(\lambda; \cos\theta)}{\mathrm{d}\theta} = \frac{1}{S_{N-1}} \tag{3.16}$$

(cf the footnote to equation (2.12)). From what has been said above, $F^{(N)}(\lambda; \cos\theta)$ may be determined uniquely. Indeed, changing in equation (3.15) the independent variable to

$$x = \cos\theta \tag{3.17}$$

yields

$$\left[(1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - Nx\frac{\mathrm{d}}{\mathrm{d}x} + \lambda(\lambda + N - 1)\right] F^{(N)}(\lambda; x) = 0 \qquad (-1 \leqslant x < 1). \tag{3.18}$$

This is a particular case of the Gegenbauer differential equation (A.12) and the requirement of regularity of $F^{(N)}(\lambda; x)$ in the interval $-1 \leqslant x < 1$, including the point $x = -1$, leads to the inference that $F^{(N)}(\lambda; x)$ may be expressed in terms of the Gegenbauer function of the first kind (cf the appendix) as

$$F^{(N)}(\lambda; x) = A^{(N)}(\lambda) C_\lambda^{((N-1)/2)}(-x) \tag{3.19}$$

where $A^{(N)}(\lambda)$ is a constant which remains to be determined. Returning to the angular variable θ, plugging equation (3.19) into the constraint (3.16) and making the limiting passage with the aid of equation (A.11) yields

$$A^{(N)}(\lambda) = \frac{\pi}{(N - 1)S_N \sin(\pi\lambda)}. \tag{3.20}$$

On combining equations (3.5), (3.19) and (3.20), we arrive at

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{\pi}{(N - 1)S_N \sin(\pi\lambda)} C_\lambda^{((N-1)/2)}(-\boldsymbol{n} \cdot \boldsymbol{n}') \qquad (N \geqslant 2). \tag{3.21}$$

In the particular case $N = 2$, the Gegenbauer function in equation (3.21) becomes the Legendre function of the first kind (cf equations (A.21) and (A.19)) and equation (3.21) yields the well-known [32–42] result for the Helmholtz Green's function on $\mathbb{S}^2$:

$$G^{(2)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\sin(\pi\lambda)} P_\lambda(-\boldsymbol{n} \cdot \boldsymbol{n}'). \tag{3.22}$$

3.4. Unification

In sections 3.2 and 3.3, we have shown that in the cases $N = 1$ and $N \geqslant 2$ the Green's function for the Helmholtz operator (2.6) is given by equations (3.12) and (3.21), respectively. However, it is easy to see that, in view of the defining relation (A.2), equations (3.12) and (3.21) may be unified into a single formula

$$G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}') = \frac{\pi}{(N-1)S_N \sin(\pi\lambda)} C_\lambda^{((N-1)/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{3.23}$$

This is the main result of section 3. After the notational changes mentioned below equation (3.4), the formula in equation (3.23) should replace the one in equation (5.1) in [22].

4. The generalized Green's function for the Helmholtz operator on $\mathbb{S}^N$

4.1. Compact closed form of the generalized Green's function for the Helmholtz operator on $\mathbb{S}^N$

It is seen from the spectral expansion (3.2) that the Green's function $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$ for the Helmholtz operator (2.6) fails to exist when

$$\lambda(\lambda + N - 1) = L(L + N - 1) \qquad (L \in \mathbb{N}). \tag{4.1}$$

In this case one may still consider the generalized Green's function $\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}')$, defined formally as a solution to the inhomogeneous Helmholtz equation

$$\left[\nabla_{\boldsymbol{n}}^2 + L(L + N - 1)\right]\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}') = \delta^{(N)}(\boldsymbol{n} - \boldsymbol{n}') - \sum_{M=1}^{d_L^{(N)}} Y_{LM}^{(N)}(\boldsymbol{n}) Y_{LM}^{(N)*}(\boldsymbol{n}') \tag{4.2}$$

subject to the orthogonality constraint

$$\oint_{\mathbb{S}^N} \mathrm{d}^N\boldsymbol{n}\, Y_{LM}^{(N)*}(\boldsymbol{n})\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}') = 0 \qquad \left(M \in \left\{1, 2, \ldots, d_L^{(N)}\right\}\right). \tag{4.3}$$

Exploiting the closure relation (2.11) and the orthonormality property (2.10), it is easy to see that the expansion of $\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}')$ in the basis of the hyperspherical harmonics is

$$\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}') = \sum_{\substack{l=0 \\ (l \neq L)}}^{\infty} \sum_{m=1}^{d_l^{(N)}} \frac{Y_{lm}^{(N)}(\boldsymbol{n}) Y_{lm}^{(N)*}(\boldsymbol{n}')}{L(L + N - 1) - l(l + N - 1)}. \tag{4.4}$$

On making use of the expansions (4.4) and (3.2), it may be verified that the following limiting relation holds:

$$\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}') = \lim_{\lambda(\lambda+N-1)\to L(L+N-1)} \frac{\partial}{\partial[\lambda(\lambda + N - 1)]} \times \{[\lambda(\lambda + N - 1) - L(L + N - 1)]G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')\}. \tag{4.5}$$

Combining this result with the closed representation (3.23) of $G^{(N)}(\lambda; \boldsymbol{n}, \boldsymbol{n}')$, after obvious transformations one finds

$$\bar{G}_L^{(N)}(\boldsymbol{n}, \boldsymbol{n}') = \frac{\pi}{(N-1)S_N} \lim_{\lambda\to L} \frac{1}{2\lambda + N - 1} \frac{\partial}{\partial\lambda} \frac{(\lambda - L)(\lambda + L + N - 1)}{\sin(\pi\lambda)} C_\lambda^{((N-1)/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.6}$$

The limiting passage in equation (4.6) is easily accomplished with the aid of the l'Hospital rule. This results in the following compact expression for the generalized Green's function for the Helmholtz operator on $\mathbb{S}^N$:

$$\bar{G}_L^{(N)}(\boldsymbol{n},\boldsymbol{n}') = \frac{(-)^L}{(N-1)S_N} \frac{\partial C_\lambda^{((N-1)/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\bigg|_{\lambda=L} + \frac{1}{(N-1)(2L+N-1)S_N} C_L^{((N-1)/2)}(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{4.7}$$

valid except for the case when it holds simultaneously that $N = 1$ *and* $L = 0$; this exceptional event will be considered in section 4.2, where we shall look at the case of $N = 1$ in detail.

4.2. *The case of* $N = 1$

In this particular case, in virtue of the defining equation (A.2), the result (4.7) simplifies to

$$\bar{G}_L^{(1)}(\boldsymbol{n},\boldsymbol{n}') = \frac{(-)^L}{4\pi} \frac{\partial C_\lambda^{(0)}(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\bigg|_{\lambda=L} + \frac{1}{8\pi} C_L^{(0)}(\boldsymbol{n}\cdot\boldsymbol{n}') \qquad (L \neq 0). \tag{4.8}$$

On exploiting the relation (A.20), linking the Gegenbauer and the Chebyshev functions of the first kinds, equation (4.8) may be cast into the form

$$\bar{G}_L^{(1)}(\boldsymbol{n},\boldsymbol{n}') = \frac{(-)^L}{2\pi L} \frac{\partial T_\lambda(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\bigg|_{\lambda=L} - \frac{1}{4\pi L^2} T_L(\boldsymbol{n}\cdot\boldsymbol{n}') \qquad (L \neq 0). \tag{4.9}$$

However, it follows from equations (A.16) and (A.17) that

$$\frac{\partial T_\lambda(x)}{\partial\lambda} = -U_{\lambda-1}(x)\sqrt{1-x^2}\arccos x \tag{4.10}$$

where $U_\lambda(x)$ is the Chebyshev function of the second kind, and consequently we find that if $L \neq 0$, then the explicit expression for the generalized Helmholtz Green's function $\bar{G}_L^{(1)}(\boldsymbol{n},\boldsymbol{n}')$ is

$$\bar{G}_L^{(1)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{2\pi L} U_{L-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\sqrt{1-(\boldsymbol{n}\cdot\boldsymbol{n}')^2}\arccos(-\boldsymbol{n}\cdot\boldsymbol{n}') - \frac{1}{4\pi L^2} T_L(\boldsymbol{n}\cdot\boldsymbol{n}') \qquad (L \neq 0). \tag{4.11}$$

To study the case of $L = 0$, we use the unconstrained formula (4.6), which, in virtue of equations (A.2) and (A.20), for $N = 1$ and $L = 0$ becomes

$$\bar{G}_L^{(1)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4}\lim_{\lambda\to 0} \frac{1}{\lambda}\frac{\partial}{\partial\lambda}\frac{\lambda}{\sin(\pi\lambda)} T_\lambda(-\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.12}$$

From this, with no difficulty we find that the explicit expression for the generalized Helmholtz Green's function $\bar{G}_0^{(1)}(\boldsymbol{n},\boldsymbol{n}')$ is

$$\bar{G}_0^{(1)}(\boldsymbol{n},\boldsymbol{n}') = -\frac{1}{4\pi}[\arccos(-\boldsymbol{n}\cdot\boldsymbol{n}')]^2 + \frac{\pi}{12}. \tag{4.13}$$

4.3. *The case of* $N = 2$

This particular case has been studied in detail by the present author in [42]. Therefore, here we shall summarize only the main results of that study.

The generalized Green's function $\bar{G}_L^{(2)}(\boldsymbol{n},\boldsymbol{n}')$ may be expressed as

$$\bar{G}_L^{(2)}(\boldsymbol{n},\boldsymbol{n}') = \frac{(-)^L}{4\pi} \frac{\partial P_\lambda(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\bigg|_{\lambda=L} + \frac{1}{4\pi(2L+1)} P_L(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{4.14}$$

where $P_\lambda(x)$ is the Legendre function of the first kind and $P_L(x)$ is the Legendre polynomial (cf the appendix); equation (4.14) also follows from equations (4.7), (2.4) and (A.21). It can be shown [42–44] that the explicit representation of the derivative $[\partial P_\lambda(x)/\partial\lambda]_{\lambda=L}$ is

$$\left.\frac{\partial P_\lambda(x)}{\partial\lambda}\right|_{\lambda=L} = P_L(x)\ln\frac{1+x}{2} + 2\sum_{l=0}^{L-1}(-)^{L+l}\frac{2l+1}{(L-l)(L+l+1)}[P_l(x) - P_L(x)]. \tag{4.15}$$

(Here and below we adopt the convention which states that if the upper limit of a summation index is less by unity than the lower one, then the sum vanishes identically.) Combining equations (4.14) and (4.15), one finds

$$\bar{G}_L^{(2)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}P_L(\boldsymbol{n}\cdot\boldsymbol{n}')\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + \frac{1}{2\pi}\sum_{l=0}^{L-1}\frac{2l+1}{(L-l)(L+l+1)}P_l(\boldsymbol{n}\cdot\boldsymbol{n}')$$
$$+\frac{1}{4\pi}\left(\frac{1}{2L+1} - 2\sum_{l=0}^{L-1}(-)^{L+l}\frac{2l+1}{(L-l)(L+l+1)}\right)P_L(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{4.16}$$

or equivalently

$$\bar{G}_L^{(2)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{4\pi}P_L(\boldsymbol{n}\cdot\boldsymbol{n}')\ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} + \frac{1}{2\pi}\sum_{l=0}^{L-1}\frac{2l+1}{(L-l)(L+l+1)}P_l(\boldsymbol{n}\cdot\boldsymbol{n}')$$
$$+\frac{1}{4\pi}[\psi(2L+2)+\psi(2L+1)-2\psi(L+1)]P_L(\boldsymbol{n}\cdot\boldsymbol{n}') \tag{4.17}$$

where

$$\psi(z) = \frac{1}{\Gamma(z)}\frac{\mathrm{d}\Gamma(z)}{\mathrm{d}z} \tag{4.18}$$

is the digamma function [45]. The identity

$$\sum_{l=0}^{L-1}(-)^{L+l+1}\frac{2l+1}{(L-l)(L+l+1)} = \psi(2L+1) - \psi(L+1) \tag{4.19}$$

which has been used to pass from equation (4.16) to equation (4.17) is easily provable with the aid of the known formula [45]

$$\psi(L+1) = -\gamma + \sum_{l=1}^{L}\frac{1}{l} \tag{4.20}$$

with γ denoting the Euler–Mascheroni constant.

4.4. The case of $N = 3$

This is the last particular case we wish to consider. From equations (4.7), (2.4) and (A.22) we have

$$\bar{G}_L^{(3)}(\boldsymbol{n},\boldsymbol{n}') = \frac{(-)^L}{4\pi^2}\left.\frac{\partial U_\lambda(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\right|_{\lambda=L} + \frac{1}{8\pi^2(L+1)}U_L(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.21}$$

However, on making use of the definitions (A.17) and (A.16) it may be easily shown that

$$\frac{\partial U_\lambda(x)}{\partial\lambda} = T_{\lambda+1}(x)\frac{\arccos x}{\sqrt{1-x^2}}. \tag{4.22}$$

Combining equations (4.21) and (4.22) yields the following final result for the generalized Helmholtz Green's function in the case considered here:

$$\bar{G}_L^{(3)}(\boldsymbol{n},\boldsymbol{n}') = -\frac{1}{4\pi^2}T_{L+1}(\boldsymbol{n}\cdot\boldsymbol{n}')\frac{\arccos(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\sqrt{1-(\boldsymbol{n}\cdot\boldsymbol{n}')^2}} + \frac{1}{8\pi^2(L+1)}U_L(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.23}$$

4.5. The general case of N odd

Having discussed in sections 4.2 and 4.4 the particular cases $N = 1$ and $N = 3$, now we proceed to analysing the case of N arbitrary odd:

$$N = 2n + 1 \qquad (n \in \mathbb{N}). \tag{4.24}$$

The only restriction imposed in what follows is that $n = 0$ and $L = 0$ cannot hold simultaneously (note, however, that this exceptional case has been already discussed at the end of section 4.2).

Under the constraint (4.24), equation (4.7) becomes

$$\bar{G}_L^{(2n+1)}(\boldsymbol{n}, \boldsymbol{n}') = (-)^L \frac{1}{2n S_{2n+1}} \frac{\partial C_\lambda^{(n)}(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\bigg|_{\lambda=L} + \frac{1}{2n(L+n)S_{2n+1}} C_L^{(n)}(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.25}$$

It follows from equation (A.15) that the Gegenbauer function $C_\lambda^{(n)}(x)$ may be expressed as

$$C_\lambda^{(n)}(x) = \frac{1}{2^n(n-1)!} \frac{\mathrm{d}^n C_{\lambda+n}^{(0)}(x)}{\mathrm{d}x^n} \tag{4.26}$$

or, in virtue of equation (A.20), equivalently as

$$C_\lambda^{(n)}(x) = \frac{1}{2^{n-1}(n-1)!(\lambda+n)} \frac{\mathrm{d}^n T_{\lambda+n}(x)}{\mathrm{d}x^n}. \tag{4.27}$$

Hence, we infer that the derivative $\partial C_\lambda^{(n)}(x)/\partial\lambda$ may be written as

$$\frac{\partial C_\lambda^{(n)}(x)}{\partial\lambda} = \frac{1}{2^{n-1}(n-1)!(\lambda+n)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{\partial T_{\lambda+n}(x)}{\partial\lambda} - \frac{1}{\lambda+n} C_\lambda^{(n)}(x). \tag{4.28}$$

Making here use of the earlier result (4.10) and of equation (A.22), performing then the n-fold differentiation with respect to x with the aid of the Leibniz theorem, and exploiting equation (A.14) we find

$$\frac{\partial C_\lambda^{(n)}(x)}{\partial\lambda} = -\frac{2n}{\lambda+n} \sum_{k=0}^{n} \frac{1}{2^k k!} C_{\lambda+k-1}^{(n-k+1)}(x) X_k(x) - \frac{1}{\lambda+n} C_\lambda^{(n)}(x) \tag{4.29}$$

where we define

$$X_k(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} \sqrt{1-x^2} \arccos x. \tag{4.30}$$

The function $X_k(x)$ may be easily shown to be of the form

$$X_k(x) = \frac{A_k(x)}{(1-x^2)^{k-1/2}} \arccos x + \frac{B_k(x)}{(1-x^2)^{k-1}} \tag{4.31}$$

where $A_k(x)$ and $B_k(x)$ are polynomials obeying the differential-difference relations

$$A_{k+1}(x) = (1-x^2)\frac{\mathrm{d}A_k(x)}{\mathrm{d}x} + (2k-1)x A_k(x) \tag{4.32a}$$

and

$$B_{k+1}(x) = (1-x^2)\frac{\mathrm{d}B_k(x)}{\mathrm{d}x} + 2(k-1)x B_k(x) - A_k(x) \tag{4.32b}$$

subject to the initial conditions

$$A_0(x) = 1 \qquad B_0(x) = 0. \tag{4.33}$$

On combining equations (4.25) and (4.29), we arrive at the final result

$$\bar{G}_L^{(2n+1)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{(L+n)S_{2n+1}}\sum_{k=0}^{n}\frac{(-)^k}{2^k k!}C_{L+k-1}^{(n-k+1)}(\boldsymbol{n}\cdot\boldsymbol{n}')X_k(-\boldsymbol{n}\cdot\boldsymbol{n}') - \frac{1}{4n(L+n)S_{2n+1}}C_L^{(n)}(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.34}$$

Exploiting equations (A.2), (A.20), (A.22), (4.31) and (4.33), it is not difficult to verify that for $n = 0$ equation (4.34) goes over into equation (4.11). Similarly, if $n = 1$, the use of equations (4.31)–(4.33), (A.13), (A.22) and (A.18) transforms equation (4.34) into equation (4.23).

4.6. The general case of N even

Finally, we proceed to considering the case when N is arbitrary even:

$$N = 2n+2 \qquad (n \in \mathbb{N}). \tag{4.35}$$

In this case, equation (4.7) becomes

$$\bar{G}_L^{(2n+2)}(\boldsymbol{n},\boldsymbol{n}') = (-)^L\frac{1}{(2n+1)S_{2n+2}}\left.\frac{\partial C_\lambda^{(n+1/2)}(-\boldsymbol{n}\cdot\boldsymbol{n}')}{\partial\lambda}\right|_{\lambda=L} + \frac{1}{(2n+1)(2L+2n+1)S_{2n+2}}C_L^{(n+1/2)}(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.36}$$

To transform the expression on the right-hand side of equation (4.36) to a more tractable form, we have to evaluate the derivative $\left[\partial C_\lambda^{(n+1/2)}(x)/\partial\lambda\right]_{\lambda=L}$. To this end, we first observe that equation (A.14) gives

$$C_\lambda^{(n+1/2)}(x) = \frac{1}{(2n-1)!!}\frac{\mathrm{d}^n C_{\lambda+n}^{(1/2)}(x)}{\mathrm{d}x^n} \tag{4.37}$$

hence, in virtue of the relationship (A.21), we have

$$C_\lambda^{(n+1/2)}(x) = \frac{1}{(2n-1)!!}\frac{\mathrm{d}^n P_{\lambda+n}(x)}{\mathrm{d}x^n} \tag{4.38}$$

and further

$$\left.\frac{\partial C_\lambda^{(n+1/2)}(x)}{\partial\lambda}\right|_{\lambda=L} = \frac{1}{(2n-1)!!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left.\frac{\partial P_\lambda(x)}{\partial\lambda}\right|_{\lambda=L+n}. \tag{4.39}$$

On inserting here our finding (4.15), after subsequent use of equations (4.19), (A.21) and (A.14), we arrive at

$$\begin{aligned}\left.\frac{\partial C_\lambda^{(n+1/2)}(x)}{\partial\lambda}\right|_{\lambda=L} &= C_L^{(n+1/2)}(x)\ln\frac{1+x}{2} \\ &\quad - \frac{n!}{(2n-1)!!}\sum_{k=1}^{n}(-)^k\frac{(2n-2k-1)!!}{k(n-k)!}\frac{C_{L+k}^{(n-k+1/2)}(x)}{(1+x)^k} \\ &\quad + 2\sum_{l=0}^{L-1}(-)^{L+l}\frac{2l+2n+1}{(L-l)(L+2n+l+1)}C_l^{(n+1/2)}(x) \\ &\quad + 2[\psi(2L+2n+1)-\psi(L+n+1)]C_L^{(n+1/2)}(x).\end{aligned} \tag{4.40}$$

With equation (4.40) in hand, we may finally rewrite the generalized Green's function (4.36) as

$$\bar{G}_L^{(2n+2)}(\boldsymbol{n},\boldsymbol{n}') = \frac{1}{(2n+1)S_{2n+2}} C_L^{(n+1/2)}(\boldsymbol{n}\cdot\boldsymbol{n}') \ln\frac{1-\boldsymbol{n}\cdot\boldsymbol{n}'}{2} - \frac{n!}{(2n+1)!!S_{2n+2}} \sum_{k=1}^{n} \frac{(2n-2k-1)!!}{k(n-k)!} \frac{C_{L+k}^{(n-k+1/2)}(\boldsymbol{n}\cdot\boldsymbol{n}')}{(1-\boldsymbol{n}\cdot\boldsymbol{n}')^k} + \frac{2}{(2n+1)S_{2n+2}} \sum_{l=0}^{L-1} \frac{2l+2n+1}{(L-l)(L+2n+l+1)} C_l^{(n+1/2)}(\boldsymbol{n}\cdot\boldsymbol{n}') + \frac{1}{(2n+1)S_{2n+2}} [\psi(2L+2n+2)+\psi(2L+2n+1) - 2\psi(L+n+1)] C_L^{(n+1/2)}(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{4.41}$$

It is easy to check that particularizing equation (4.41) to the case $n = 0$ yields the result given in equation (4.17).

Appendix. Gegenbauer, Legendre and Chebyshev functions and polynomials

In this appendix, we summarize these properties of the Gegenbauer, Legendre and Chebyshev functions and polynomials, which have proved to be useful in the considerations carried out in sections 3 and 4. The presented formulae have been excerpted from [24, 45–47].

Throughout the whole appendix, x is a real variable from the interval $-1 \leqslant x \leqslant 1$.

For $\alpha \neq 0$, the Gegenbauer function (of the first kind) is defined as

$$C_\lambda^{(\alpha)}(x) = \frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)\Gamma(2\alpha)}\, {}_2F_1\left(-\lambda, \lambda+2\alpha; \alpha+\frac{1}{2}; \frac{1-x}{2}\right) \tag{A.1}$$

where ${}_2F_1$ is the Gauss hypergeometric series. For $\alpha = 0$ and $\lambda \neq 0$, it is defined through the limiting procedure

$$C_\lambda^{(0)}(x) = \lim_{\alpha\to 0} \frac{1}{\alpha} C_\lambda^{(\alpha)}(x) \tag{A.2}$$

so that

$$C_\lambda^{(0)}(x) = \frac{2}{\lambda}\, {}_2F_1\left(-\lambda, \lambda; \frac{1}{2}; \frac{1-x}{2}\right) = \frac{2}{\lambda}\cos(\lambda \arccos x) \tag{A.3}$$

while for $\alpha = 0$ and $\lambda = 0$, by definition, one has

$$C_0^{(0)}(x) = 1. \tag{A.4}$$

When $\lambda = L \in \mathbb{N}$, the function $C_\lambda^{(\alpha)}(x)$ degenerates to the Gegenbauer (ultraspherical) polynomial. If $\alpha \neq 0$, this polynomial is given by the Rodrigues-type formula

$$C_L^{(\alpha)}(x) = (-)^L \frac{\sqrt{\pi}}{2^{L+2\alpha-1}L!} \frac{\Gamma(L+2\alpha)}{\Gamma(\alpha)\Gamma\left(L+\alpha+\frac{1}{2}\right)} (1-x^2)^{-\alpha+1/2} \frac{\mathrm{d}^L}{\mathrm{d}x^L}(1-x^2)^{L+\alpha-1/2}. \tag{A.5}$$

If $\alpha = 0$ and $L \neq 0$, from equations (A.2) and (A.5) one finds

$$C_L^{(0)}(x) = (-)^L \frac{2}{L(2L-1)!!} \sqrt{1-x^2} \frac{\mathrm{d}^L}{\mathrm{d}x^L}(1-x^2)^{L-1/2}. \tag{A.6}$$

The case of $\alpha = 0$ and $L = 0$ is covered by equation (A.4). The Gegenbauer polynomials have a definite parity with respect to the reflection at $x = 0$; one has

$$C_L^{(\alpha)}(-x) = (-)^L C_L^{(\alpha)}(x). \tag{A.7}$$

For $\alpha \neq 0$ it holds that[2]

$$C_{\lambda}^{(\alpha)}(x) = \frac{\sqrt{\pi}}{2^{\alpha-1/2}} \frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)\Gamma(\alpha)} (1-x^2)^{-\alpha/2+1/4} P_{\lambda+\alpha-1/2}^{(-\alpha+1/2)}(x) \tag{A.8}$$

where

$$\begin{aligned} P_{\nu}^{(\mu)}(x) &= \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right) \\ &= \frac{2^{\mu}}{\Gamma(1-\mu)} (1-x^2)^{-\mu/2} {}_2F_1\left(-\nu-\mu, \nu+1-\mu; 1-\mu; \frac{1-x}{2}\right) \end{aligned} \tag{A.9}$$

is the associated Legendre function of the first kind. From the relationship (A.8) and from the following limiting representations

$$P_{\nu}^{(\mu)}(x) \xrightarrow{x\downarrow -1} \begin{cases} \dfrac{\Gamma(-\mu)}{2^{\mu/2}\Gamma(-\nu-\mu)\Gamma(\nu+1-\mu)}(1+x)^{\mu/2} & \text{for} \quad \mu < 0 \\ \dfrac{\sin(\pi\nu)}{\pi} \ln(1+x) & \text{for} \quad \mu = 0, \end{cases} \tag{A.10}$$

it may be inferred that

$$C_{\lambda}^{(\alpha)}(x) \xrightarrow{x\downarrow -1} \begin{cases} -\dfrac{\sin(\pi\lambda)}{\sqrt{\pi}} \dfrac{\Gamma\left(\alpha-\frac{1}{2}\right)}{2^{\alpha-1/2}\Gamma(\alpha)}(1+x)^{-\alpha+1/2} & \text{for} \quad \alpha > \dfrac{1}{2} \\ \dfrac{\sin(\pi\lambda)}{\pi} \ln(1+x) & \text{for} \quad \alpha = \dfrac{1}{2}. \end{cases} \tag{A.11}$$

The Gegenbauer functions $C_{\lambda}^{(\alpha)}(x)$ and $C_{\lambda}^{(\alpha)}(-x)$ satisfy the second-order linear differential equation

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (2\alpha+1)x\frac{\mathrm{d}}{\mathrm{d}x} + \lambda(\lambda+2\alpha)\right] F(x) = 0. \tag{A.12}$$

If $\alpha \geqslant \frac{1}{2}$ and $\lambda \notin \mathbb{N}$, the function $C_{\lambda}^{(\alpha)}(-x)$ is the only (up to a multiplicative factor) solution to the above equation which remains finite at $x = -1$.

Among useful properties of the Gegenbauer functions there are: the recurrence relation

$$2\alpha(1-x^2)C_{\lambda-1}^{(\alpha+1)}(x) = (\lambda+2\alpha)xC_{\lambda}^{(\alpha)}(x) - (\lambda+1)C_{\lambda+1}^{(\alpha)}(x) \tag{A.13}$$

and the differential relations

$$\frac{\mathrm{d}^n C_{\lambda}^{(\alpha)}(x)}{\mathrm{d}x^n} = 2^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} C_{\lambda-n}^{(\alpha+n)}(x) \qquad (\alpha \neq 0) \tag{A.14}$$

$$\frac{\mathrm{d}^n C_{\lambda}^{(0)}(x)}{\mathrm{d}x^n} = 2^n (n-1)! C_{\lambda-n}^{(n)}(x). \tag{A.15}$$

The Chebyshev functions of the first and the second kinds are defined as

$$T_{\lambda}(x) = \cos(\lambda \arccos x) \tag{A.16}$$

and

$$U_{\lambda}(x) = \frac{\sin[(\lambda+1)\arccos x]}{\sin(\arccos x)} = \frac{\sin[(\lambda+1)\arccos x]}{\sqrt{1-x^2}}, \tag{A.17}$$

[2] Equation (A.8) does not contradict equations (10.9.33) in [24] and (3.15.4) in [46] since the definition of the associated Legendre function used therein differs by a phase factor from ours.

respectively. A relationship between the Chebyshev functions is

$$U_{\lambda+1}(x) - xU_\lambda(x) = T_{\lambda+1}(x). \tag{A.18}$$

The Legendre function of the first kind is defined as

$$P_\lambda(x) = {}_2F_1\left(-\lambda, \lambda+1; 1; \frac{1-x}{2}\right). \tag{A.19}$$

All these three functions are simply related to the Gegenbauer functions; it holds that

$$T_\lambda(x) = \begin{cases} \frac{1}{2}\lambda C_\lambda^{(0)}(x) & \text{for} \quad \lambda \neq 0 \\ C_0^{(0)}(x) & \text{for} \quad \lambda = 0 \end{cases} \tag{A.20}$$

$$P_\lambda(x) = C_\lambda^{(1/2)}(x) \tag{A.21}$$

and

$$U_\lambda(x) = C_\lambda^{(1)}(x). \tag{A.22}$$

When $\lambda = L \in \mathbb{N}$, the functions in question degenerate to polynomials named after their parent functions and given by the Rodrigues-type formulae

$$T_L(x) = (-)^L \frac{1}{(2L-1)!!} \sqrt{1-x^2} \frac{\mathrm{d}^L}{\mathrm{d}x^L} (1-x^2)^{L-1/2} \tag{A.23}$$

$$P_L(x) = \frac{1}{2^L L!} \frac{\mathrm{d}^L}{\mathrm{d}x^L} (x^2-1)^L \tag{A.24}$$

and

$$U_L(x) = (-)^L \frac{L+1}{(2L+1)!!} \frac{1}{\sqrt{1-x^2}} \frac{\mathrm{d}^L}{\mathrm{d}x^L} (1-x^2)^{L+1/2}, \tag{A.25}$$

respectively.

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