IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **44** (2011) 065203 (14pp) doi:10.1088/1751-8113/44/6/065203

Green's function for the wavized Maxwell fish-eye problem

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Received 29 September 2010, in final form 6 December 2010
Published 12 January 2011

Abstract
Unique transformation properties under the hyperspherical inversion of a partial differential equation describing a stationary scalar wave in an N-dimensional ($N \geqslant 2$) Maxwell fish-eye medium are exploited to construct a closed form of Green's function for that equation. For those wave numbers for which Green's function fails to exist, the generalized Green's function is derived. Prospective physical applications are mentioned.

PACS numbers: 02.30.Jr, 02.30.Gp, 42.25.Bs, 42.79.Ry
Mathematics Subject Classification: 35J08, 78A10

1. Introduction

In 1854, Maxwell [1] pointed at a remarkable property of an infinite optical medium with the refraction index

$$n_{\mathrm{fe}}(\boldsymbol{r}) = \frac{2n_0\rho^2}{r^2+\rho^2} \qquad (\rho > 0). \tag{1.1}$$

Within the framework of the geometrical optics, he proved that paths of all rays emitted from an arbitrarily located point $\boldsymbol{r}'$ are circles having two points in common: the source point $\boldsymbol{r}'$ and the image point $-\rho^2\boldsymbol{r}'/r'^2$. For this medium Maxwell coined the name 'the fish-eye'. In 1926, Carathéodory [2] observed that there is a geometrical correspondence between the circular rays in the fish-eye medium and the geodesics on a sphere. This geometric thread was pursued further by other researchers within the group-theoretical framework (cf, e.g., [3]).

It seems that Demkov and Ostrovsky [4] were the first to discuss the wavized scalar fish-eye problem. Specifically, they considered the equation

$$\left[\nabla^2 + \frac{4\nu(\nu+1)\rho^2}{(r^2+\rho^2)^2}\right]\Psi(\boldsymbol{r}) = 0 \qquad (\rho > 0) \tag{1.2}$$

in $\mathbb{R}^3$, subject to the boundary condition that $\Psi(\boldsymbol{r})$ vanishes at infinity. They proceeded in two directions. First, they solved analytically a spectral problem with ν being an eigenparameter and showed that the resulting spectrum is purely discrete and eigenfunctions may be expressed in terms of the Gegenbauer polynomials. Second, they proved that equation (1.2) possesses a certain remarkable transformation property under the geometrical inversion in a certain class of spheres and ingeniously exploited this fact to construct a closed form of the relevant Green's function in $\mathbb{R}^3$. Group-theoretical properties of the scalar fish-eye wave equation (1.2) were then investigated in $\mathbb{R}^2$ by Frank *et al* [5, 6]. Lately, the two-dimensional fish-eye medium has been studied by Makowski and Górska [7] in the context of the construction of pertinent coherent states. Finally, in two very recent papers, Leonhardt [8] and Leonhardt and Philbin [9] have argued that the geometric-optical perfect focusing property of the fish-eye medium, discovered by Maxwell, holds as well within the wave-optics framework; that issue will be critically reexamined in our upcoming work, with the aid of the results presented below.

This paper is the first out of a series of several reports in which we shall expose results of our research on the wave properties of the Maxwell fish-eye and related media. Here, we derive a closed form of Green's function for the scalar fish-eye wave equation in $\mathbb{R}^N$, $N \geqslant 2$. The particular method we employ generalizes the aforementioned one used by Demkov and Ostrovsky in the case of $N = 3$ and exploits a peculiar transformation property of the N-dimensional fish-eye equation under the hyperspherical inversion.

The structure of the paper is as follows. In section 2, we investigate transformation properties of a class of partial differential equations under the hyperspherical inversion, with a special focus on the N-dimensional fish-eye equation. The results of that investigation are used in section 3 to construct Green's function for the fish-eye problem. The case when Green's function fails to exist and is to be replaced by the generalized Green's function is considered in section 4. Prospective physical applications of the results are briefly discussed in section 5. The paper ends with an appendix, in which a number of closed-form representations of the derivative $[\partial P_\nu^{-N/2+1}(x)/\partial\nu]_{\nu=n+N/2-1}$, with $x \in (-1, 1)$, $N \in \mathbb{N}\setminus\{0, 1\}$ and $n \in \mathbb{N}$, required in section 4, are displayed. The list of references attached has been intended to contain representative items rather than to be a comprehensive one. An exhaustive listing of works relevant to the Maxwell fish-eye problem will be included in one of our forthcoming papers.

2. Transformation properties of a class of partial differential equations under the hyperspherical inversion

At first, we establish the following

Lemma 1. *If* $\Psi(\boldsymbol{r})$ *(with* $\boldsymbol{r} \in \mathbb{R}^N$*,* $N \geqslant 2$*) satisfies the equation*

$$[\boldsymbol{\nabla}^2 + k^2 n^2(\boldsymbol{r})]\Psi(\boldsymbol{r}) = 0, \tag{2.1}$$

then for arbitrary $R \in \mathbb{R}$ *and* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$ *it holds that*

$$\left[\boldsymbol{\nabla}^2 + \frac{k^2 R^4}{|\boldsymbol{r}-\boldsymbol{a}|^4} n^2\left(\frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a}) + \boldsymbol{b}\right)\right] \frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}} \Psi\left(\frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a}) + \boldsymbol{b}\right) = 0. \tag{2.2}$$

Proof. The transformation

$$\boldsymbol{r} \mapsto \frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a}) + \boldsymbol{b} \tag{2.3}$$

results in the following alteration of the infinitesimal line element:

$$(\mathrm{d}\boldsymbol{r})^2 \mapsto \frac{R^4}{|\boldsymbol{r}-\boldsymbol{a}|^4}(\mathrm{d}\boldsymbol{r})^2. \tag{2.4}$$

Hence, substitution (2.3) implies the following transformation of the N-dimensional Laplace operator:

$$\nabla^2 \mapsto \frac{|\boldsymbol{r}-\boldsymbol{a}|^{2N}}{R^{2N}}\nabla\cdot\left(\frac{R^{2(N-2)}}{|\boldsymbol{r}-\boldsymbol{a}|^{2(N-2)}}\nabla\right). \tag{2.5}$$

Using the easily provable differential identity

$$\nabla\cdot\left(\frac{R^{2(N-2)}}{|\boldsymbol{r}-\boldsymbol{a}|^{2(N-2)}}\nabla\right) = \frac{R^{2(N-2)}}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}}\nabla^2\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}} - \frac{R^{2(N-2)}}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}}\left(\nabla^2\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}}\right), \tag{2.6}$$

and exploiting the fact that the function $1/|\boldsymbol{r}-\boldsymbol{a}|^{N-2}$ is harmonic in $\mathbb{R}^N$, we see that transformation (2.3) changes equation (2.1) into

$$\left[\frac{|\boldsymbol{r}-\boldsymbol{a}|^{N+2}}{R^4}\nabla^2\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}} + k^2n^2\left(\frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{b}\right)\right]\Psi\left(\frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{b}\right) = 0, \tag{2.7}$$

which immediately leads to equation (2.2). □

Actually, the above lemma offers a bit more than necessary for the purposes of this paper. In view of our needs, in what follows we shall restrict ourselves to the special case when the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal. It is then evident that the resulting transformation

$$\boldsymbol{r} \mapsto \frac{R^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{a} \tag{2.8}$$

is the geometric inversion in the hypersphere of radius R centered at the point with the radius vector $\boldsymbol{r}=\boldsymbol{a}$. (In fact, if $\boldsymbol{b}=\boldsymbol{a}$ and $k=0$, the lemma is simply an N-dimensional extension of the well-known Kelvin inversion theorem for harmonic functions [10].)

Now we turn to the fish-eye problem. Application of the following special case of inversion (2.8)

$$\boldsymbol{r} \mapsto \frac{\rho^2+a^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{a} \quad \Rightarrow \quad r \mapsto \frac{a}{|\boldsymbol{r}-\boldsymbol{a}|}\left|\boldsymbol{r}+\boldsymbol{a}\frac{\rho^2}{a^2}\right| \tag{2.9}$$

to the fish-eye refraction index (1.1) gives

$$n_{\mathrm{fe}}\left(\frac{\rho^2+a^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{a}\right) = \frac{|\boldsymbol{r}-\boldsymbol{a}|^2}{\rho^2+a^2}\frac{2n_0\rho^2}{r^2+\rho^2} = \frac{|\boldsymbol{r}-\boldsymbol{a}|^2}{\rho^2+a^2}n_{\mathrm{fe}}(\boldsymbol{r}). \tag{2.10}$$

Combining this property of the index $n_{\mathrm{fe}}(\boldsymbol{r})$ with the result stated in the lemma, we arrive at

Corollary. *If the function* $\Psi(\boldsymbol{r})$ *(with* $\boldsymbol{r}\in\mathbb{R}^N$, $N\geqslant 2$*) solves the fish-eye equation*

$$\left[\nabla^2 + \frac{4n_0^2k^2\rho^4}{(r^2+\rho^2)^2}\right]\Psi(\boldsymbol{r}) = 0 \qquad (\rho>0), \tag{2.11}$$

then for arbitrary $\boldsymbol{a}\in\mathbb{R}^N$ *the function*

$$\hat{\mathcal{I}}(\boldsymbol{a},\sqrt{\rho^2+a^2})\Psi(\boldsymbol{r}) \equiv \frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}}\Psi\left(\frac{\rho^2+a^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{a}\right) \tag{2.12}$$

also solves this equation, i.e. it holds that

$$\left[\boldsymbol{\nabla}^2 + \frac{4n_0^2k^2\rho^4}{(r^2+\rho^2)^2}\right]\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}}\Psi\left(\frac{\rho^2+a^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a})+\boldsymbol{a}\right) = 0 \qquad (\rho > 0). \tag{2.13}$$

In the particular case of $N = 3$, the above result was established by Demkov and Ostrovsky [4] (see also [11]).

3. Green's function for the fish-eye problem

We are now ready to construct the N-dimensional fish-eye Green's function $G_\nu(\boldsymbol{r},\boldsymbol{r}')$. According to the general theory of Green's functions for elliptic partial differential operators, $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ is a single-valued solution to the fish-eye equation

$$\left[\boldsymbol{\nabla}^2 + \frac{4n_0^2k^2\rho^4}{(r^2+\rho^2)^2}\right]G_\nu(\boldsymbol{r},\boldsymbol{r}') = 0 \qquad (\rho > 0) \tag{3.1}$$

everywhere in $\mathbb{R}^N$ except for the source point $\boldsymbol{r} = \boldsymbol{r}'$, where it diverges according to

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') \xrightarrow{r\to r'} \frac{1}{2\pi}\ln|\boldsymbol{r}-\boldsymbol{r}'| \qquad (N = 2) \tag{3.2}$$

or

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') \xrightarrow{r\to r'} -\frac{1}{(N-2)S_{N-1}|\boldsymbol{r}-\boldsymbol{r}'|^{N-2}} \qquad (N \geqslant 3). \tag{3.3}$$

In the last equation

$$S_{N-1} = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \tag{3.4}$$

is a surface area of a unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$ embedded in $\mathbb{R}^N$. At infinity, we require

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') \xrightarrow{r\to\infty} \frac{C_\nu(\boldsymbol{r}')}{r^{N-2}} \tag{3.5}$$

(the non-zero constant $C_\nu(\boldsymbol{r}')$ appearing in condition (3.5) will be determined later). The parameter ν is defined as

$$\nu = \frac{-1+\sqrt{1+4n_0^2k^2\rho^2}}{2} \qquad (\nu \to 0 \text{ for } n_0 \to 0) \tag{3.6}$$

and reasons for its introduction will become clear shortly. In what follows, we admit that the product $n_0^2k^2$, hence also ν, may be complex.

At first, consider the case when the source is located at the center of symmetry of the medium. Evidently, the corresponding Green's function $G_\nu(\boldsymbol{r},\boldsymbol{0})$ must be spherically symmetric, being a function of $r = |\boldsymbol{r}|$ only. Hence, it follows that $G_\nu(\boldsymbol{r},\boldsymbol{0})$ obeys

$$\left[\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{4\nu(\nu+1)\rho^2}{(r^2+\rho^2)^2}\right]G_\nu(\boldsymbol{r},\boldsymbol{0}) = 0 \tag{3.7}$$

except for the point $\boldsymbol{r} = \boldsymbol{0}$, where it behaves according to equations (3.2) or (3.3) with $\boldsymbol{r}' = \boldsymbol{0}$. The substitution

$$G_\nu(\boldsymbol{r},\boldsymbol{0}) = \left(\frac{\rho}{r}\right)^{N/2-1} F\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right) \tag{3.8}$$

leads to the following differential equation for the function F:

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + \nu(\nu+1) - \frac{\mu^2}{1-x^2}\right]F(x) = 0, \tag{3.9}$$

where

$$x = \frac{r^2-\rho^2}{r^2+\rho^2} \qquad (-1 \leqslant x \leqslant 1) \tag{3.10}$$

and

$$\mu = \frac{N-2}{2}. \tag{3.11}$$

Equation (3.9) is the associated Legendre equation. Its general solution, written in the form most suitable for the present purposes, is

$$F(x) = AP_\nu^{-\mu}(x) + BR_\nu^{\mu}(x), \tag{3.12}$$

with A, B being arbitrary constants and with

$$R_\nu^\mu(x) = Q_\nu^\mu(x) + \frac{\mathrm{i}\pi}{2}P_\nu^\mu(x) = \frac{\pi}{2\sin(\pi\mu)}\left[\mathrm{e}^{\mathrm{i}\pi\mu}P_\nu^\mu(x) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(x)\right]. \tag{3.13}$$

Here, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are the associated Legendre functions (on the cut $-1 \leqslant x \leqslant 1$) of the first and second kinds, respectively (occasionally, $R_\nu^\mu(x)$ is called the associated Legendre function of the third kind). The general character of solution (3.12) follows from the fact that the Wronskian of $P_\nu^{-\mu}(x)$ and $R_\nu^\mu(x)$ is

$$W[P_\nu^{-\mu}(x), R_\nu^\mu(x)] = \frac{\exp(\mathrm{i}\pi\mu)}{1-x^2}, \tag{3.14}$$

i.e. it vanishes nowhere. Henceforth, we adopt the standard convention and write $P_\nu(x)$ and $R_\nu(x)$ in place of $P_\nu^0(x)$ and $R_\nu^0(x)$.

With the general solution to equation (3.9) in hand, we see that Green's function $G_\nu(\boldsymbol{r},\mathbf{0})$ is of the form

$$G_\nu(\boldsymbol{r},\mathbf{0}) = A\left(\frac{\rho}{r}\right)^{N/2-1}P_\nu^{-N/2+1}\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right) + B\left(\frac{\rho}{r}\right)^{N/2-1}R_\nu^{N/2-1}\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right), \tag{3.15}$$

ν being defined in equation (3.6). We shall fix the values of the constants A and B in two steps. At first, we investigate the asymptotics of the expression in equation (3.15) as $r \to \infty$. Using the known formulas [12, p 196]

$$P_\nu^{-\mu}(x) \overset{x\to1-0}{\longrightarrow} \frac{1}{\Gamma(\mu+1)}\left(\frac{1-x}{2}\right)^{\mu/2} \qquad (\mu \neq -1, -2, \ldots), \tag{3.16}$$

$$R_\nu(x) \overset{x\to1-0}{\longrightarrow} -\frac{1}{2}\ln(1-x), \tag{3.17}$$

$$R_\nu^\mu(x) \overset{x\to1-0}{\longrightarrow} \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi\mu}\Gamma(\mu)\left(\frac{1-x}{2}\right)^{-\mu/2} \qquad (\mathrm{Re}\,\mu > 0), \tag{3.18}$$

we see that constraint (3.5) is fulfilled iff $B = 0$, and consequently

$$G_\nu(\boldsymbol{r},\mathbf{0}) = A\left(\frac{\rho}{r}\right)^{N/2-1}P_\nu^{-N/2+1}\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right). \tag{3.19}$$

In the second step, we investigate the asymptotics of the right-hand side of equation (3.19) as $\boldsymbol{r} \to \mathbf{0}$. Exploiting the formulas [12, p 197]

$$P_\nu(x) \overset{x\to-1+0}{\longrightarrow} \frac{\sin(\pi\nu)}{\pi}\ln(1+x) \tag{3.20}$$

and

$$P_\nu^{-\mu}(x) \overset{x\to-1+0}{\longrightarrow} \frac{\Gamma(\mu)}{\Gamma(\nu+\mu+1)\Gamma(-\nu+\mu)} \left(\frac{1+x}{2}\right)^{-\mu/2} \qquad (\operatorname{Re}\mu > 0), \tag{3.21}$$

we find that constraints (3.2) and (3.3) will be satisfied iff

$$G_\nu(\boldsymbol{r}, \boldsymbol{0}) = \frac{1}{4\sin(\pi\nu)} P_\nu\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right) \qquad (N=2) \tag{3.22}$$

and

$$G_\nu(\boldsymbol{r}, \boldsymbol{0}) = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}} \frac{P_\nu^{-N/2+1}\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right)}{(r\rho)^{N/2-1}} \qquad (N \geqslant 3), \tag{3.23}$$

respectively. Since it holds that

$$\sin(\pi\nu) = -\frac{\pi}{\Gamma(\nu+1)\Gamma(-\nu)}, \tag{3.24}$$

equations (3.22) and (3.23) may be collected into a single formula

$$G_\nu(\boldsymbol{r}, \boldsymbol{0}) = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}} \frac{P_\nu^{-N/2+1}\left(\frac{r^2-\rho^2}{r^2+\rho^2}\right)}{(r\rho)^{N/2-1}}. \tag{3.25}$$

From this, using relation (3.16), we deduce that for $\boldsymbol{r}' = \boldsymbol{0}$ the constant C_ν in the asymptotic relation (3.5) is

$$C_\nu(\boldsymbol{0}) = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}\Gamma\left(\frac{N}{2}\right)}. \tag{3.26}$$

To find Green's function for an arbitrary location of the source point $\boldsymbol{r}'$, we consider the transformed function

$$\hat{\mathcal{I}}(\boldsymbol{a}, \sqrt{\rho^2+a^2}) G_\nu(\boldsymbol{r}, \boldsymbol{0}) = \frac{1}{|\boldsymbol{r}-\boldsymbol{a}|^{N-2}} G_\nu\left(\frac{\rho^2+a^2}{|\boldsymbol{r}-\boldsymbol{a}|^2}(\boldsymbol{r}-\boldsymbol{a}) + \boldsymbol{a}, \boldsymbol{0}\right), \tag{3.27}$$

with the center of the inversion sphere (of radius $\sqrt{\rho^2+a^2}$) located at the point

$$\boldsymbol{a} = -\boldsymbol{r}'\frac{\rho^2}{r'^2}. \tag{3.28}$$

Using equation (3.25), the explicit form of this transformed function, denoted hereafter as $g_\nu(\boldsymbol{r}, \boldsymbol{r}')$, is seen to be

$$g_\nu(\boldsymbol{r}, \boldsymbol{r}') = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}} \left(\frac{r'}{\rho^3}\right)^{N/2-1} \frac{P_\nu^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|^{N/2-1}\left|\boldsymbol{r}+\boldsymbol{r}'\frac{\rho^2}{r'^2}\right|^{N/2-1}}. \tag{3.29}$$

In view of the results of section 2, we know for sure that the function in equation (3.29) solves the fish-eye equation, except, possibly, for some isolated points. It is evident that the points at which the behavior of $g_\nu(\boldsymbol{r}, \boldsymbol{r}')$ should be investigated are the two finite points $\boldsymbol{r} = \boldsymbol{r}'$ and $\boldsymbol{r} = -\boldsymbol{r}'\rho^2/r'^2$, and also the point at infinity. Using the asymptotic relations (3.20) and (3.21), we derive

$$g_\nu(\boldsymbol{r}, \boldsymbol{r}') \overset{\boldsymbol{r}\to\boldsymbol{r}'}{\longrightarrow} \frac{1}{2\pi} \ln|\boldsymbol{r}-\boldsymbol{r}'| \qquad (N=2) \tag{3.30}$$

and

$$g_\nu(\boldsymbol{r}, \boldsymbol{r}') \overset{\boldsymbol{r}\to\boldsymbol{r}'}{\longrightarrow} -\frac{1}{(N-2)S_{N-1}|\boldsymbol{r}-\boldsymbol{r}'|^{N-2}} \left(\frac{r'}{\rho^2}\right)^{N-2} \qquad (N \geqslant 3), \tag{3.31}$$

i.e. the 'inverted' function diverges for $\boldsymbol{r} \to \boldsymbol{r}'$ in the same manner (save for the factor $(r'/\rho^2)^{N-2}$ when $N \geqslant 3$) as Green's function $G_\nu(\boldsymbol{r}, \boldsymbol{r}')$ (cf equations (3.2) and (3.3)). Furthermore, it is seen that for $r \to \infty$ function (3.29) decays asymptotically as

$$g_\nu(\boldsymbol{r}, \boldsymbol{r}') \xrightarrow{r\to\infty} -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}}\left(\frac{r'}{\rho^3}\right)^{N/2-1}\frac{P_\nu^{-N/2+1}\left(\frac{\rho^2-r'^2}{\rho^2+r'^2}\right)}{r^{N-2}}, \tag{3.32}$$

i.e. in the same functional manner with r as prescribed for $G_\nu(\boldsymbol{r}, \boldsymbol{r}')$ in equation (3.5). Finally, with the help of the identity

$$-1 + \frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)} = 1 - \frac{2r'^2\left(\boldsymbol{r}+\boldsymbol{r}'\frac{\rho^2}{r'^2}\right)^2}{(r^2+\rho^2)(r'^2+\rho^2)} \tag{3.33}$$

and the asymptotic relation (3.16), we find that for $\boldsymbol{r} \to -\boldsymbol{r}'\rho^2/r'^2$ function (3.29) remains finite:

$$g_\nu(\boldsymbol{r}, \boldsymbol{r}') \xrightarrow{\boldsymbol{r}\to-\boldsymbol{r}'\rho^2/r'^2} -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}\Gamma\left(\frac{N}{2}\right)}\left(\frac{r'}{\rho}\right)^{2(N-2)}\frac{1}{(r'^2+\rho^2)^{N-2}}. \tag{3.34}$$

Thus, we see that the function $(\rho^2/r')^{N-2}g_\nu(\boldsymbol{r}, \boldsymbol{r}')$ satisfies all conditions imposed on $G_\nu(\boldsymbol{r}, \boldsymbol{r}')$ in equations (3.1)–(3.5). Hence, we conclude that the closed form of the N-dimensional fish-eye Green's function for an arbitrary location of the source point $\boldsymbol{r}'$ is

$$G_\nu(\boldsymbol{r}, \boldsymbol{r}') = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}}\left(\frac{\rho}{r'}\right)^{N/2-1}\frac{P_\nu^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|^{N/2-1}\left|\boldsymbol{r}+\boldsymbol{r}'\frac{\rho^2}{r'^2}\right|^{N/2-1}} \tag{3.35}$$

and that the constant $C_\nu(\boldsymbol{r}')$ in the asymptotic constraint (3.5) is

$$C_\nu(\boldsymbol{r}') = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}}\left(\frac{\rho}{r'}\right)^{N/2-1}P_\nu^{-N/2+1}\left(\frac{\rho^2-r'^2}{\rho^2+r'^2}\right). \tag{3.36}$$

Since the differential operator in equation (3.1) is symmetric with respect to the scalar product $\langle\chi|\phi\rangle_N \equiv \int_{\mathbb{R}^N} \mathrm{d}^N\boldsymbol{r}\, \chi(\boldsymbol{r})\phi(\boldsymbol{r})$, the fish-eye Green's function should be symmetric with respect to the interchange of the source and observation points:

$$G_\nu(\boldsymbol{r}, \boldsymbol{r}') = G_\nu(\boldsymbol{r}', \boldsymbol{r}). \tag{3.37}$$

However, the representation of $G_\nu(\boldsymbol{r}, \boldsymbol{r}')$ given in equation (3.35) does not exhibit this property explicitly. To show that nevertheless relation (3.37) is satisfied, we observe that it holds that

$$r'\left|\boldsymbol{r}+\boldsymbol{r}'\frac{\rho^2}{r'^2}\right| = r\left|\boldsymbol{r}'+\boldsymbol{r}\frac{\rho^2}{r^2}\right| = \sqrt{r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4}. \tag{3.38}$$

Consequently, Green's function (3.35) may be alternatively rewritten in either of the following two manifestly symmetric forms:

$$G_\nu(\boldsymbol{r}, \boldsymbol{r}') = -\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}}\frac{\rho^{N/2-1}P_\nu^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|^{N/2-1}(r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4)^{N/4-1/2}} \tag{3.39}$$

or

$$\begin{aligned}G_\nu(\boldsymbol{r}, \boldsymbol{r}') = &-\frac{\Gamma\left(\frac{N}{2}+\nu\right)\Gamma\left(\frac{N}{2}-\nu-1\right)}{4\pi^{N/2}}\\ &\times\frac{\rho^{N/2-1}P_\nu^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{(rr')^{N/4-1/2}|\boldsymbol{r}-\boldsymbol{r}'|^{N/2-1}\left|\boldsymbol{r}+\boldsymbol{r}'\frac{\rho^2}{r'^2}\right|^{N/4-1/2}\left|\boldsymbol{r}'+\boldsymbol{r}\frac{\rho^2}{r^2}\right|^{N/4-1/2}}.\end{aligned} \tag{3.40}$$

Still another representation of $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ displaying its symmetry is the one in terms of the Gegenbauer function of the first kind. Using the known relationship

$$C_\alpha^\lambda(x)=\frac{\sqrt{\pi}}{2^{\lambda-1/2}}\frac{\Gamma(\alpha+2\lambda)}{\Gamma(\lambda)\Gamma(\alpha+1)}(1-x^2)^{-\lambda/2+1/4}P_{\alpha+\lambda-1/2}^{-\lambda+1/2}(x)\qquad(-1\leqslant x\leqslant 1),\tag{3.41}$$

equation (3.35) is transformed into

$$G_\nu(\boldsymbol{r},\boldsymbol{r}')=\frac{2^{N-4}\Gamma\left(\frac{N-1}{2}\right)}{\pi^{(N-1)/2}\sin\left[\pi\left(\frac{N}{2}-\nu\right)\right]}\frac{\rho^{N-2}C_{\nu-N/2+1}^{(N-1)/2}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{(r^2+\rho^2)^{N/2-1}(r'^2+\rho^2)^{N/2-1}}.\tag{3.42}$$

Let us consider some particular cases. For $N=2$, from either of equations (3.35), (3.39) or (3.40), with the aid of equation (3.24), we find

$$G_\nu(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\sin(\pi\nu)}P_\nu\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)\qquad(N=2).\tag{3.43}$$

Next, it appears that the representations of $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ found above simplify greatly when N is odd, as then the Legendre function appearing in equations (3.35), (3.39) and (3.40) may be expressed in terms of trigonometric and inverse trigonometric functions either as [12, p 168]

$$\begin{aligned}P_\nu^{-N/2+1}(x)=&\frac{\left(\frac{N-3}{2}\right)!}{2^{N/2-2}\sqrt{\pi}}(1-x^2)^{-N/4+1/2}\sum_{k=0}^{(N-3)/2}(-)^k\frac{\Gamma\left(k+\nu-\frac{N}{2}+2\right)}{k!\Gamma\left(k+\nu+\frac{3}{2}\right)\left(\frac{N-3}{2}-k\right)!}\\&\times\sin\left[\left(2k+\nu-\tfrac{N}{2}+2\right)\arccos x\right]\qquad(N\text{ odd},\ N\geqslant 3)\end{aligned}\tag{3.44}$$

or as [12, p 169]

$$\begin{aligned}P_\nu^{-N/2+1}(x)=&\sqrt{\frac{2}{\pi}}\,\Gamma\left(\nu-\tfrac{N}{2}+2\right)\sum_{k=0}^{(N-3)/2}\frac{\left(k+\frac{N-3}{2}\right)!}{2^k k!\Gamma\left(k+\nu+\frac{3}{2}\right)\left(\frac{N-3}{2}-k\right)!}\\&\times\frac{\sin\left[\left(k+\nu+\frac{1}{2}\right)\arccos x+\left(k-\frac{N-3}{2}\right)\frac{\pi}{2}\right]}{(1-x^2)^{k/2+1/4}}\qquad(N\text{ odd},\ N\geqslant 3).\end{aligned}\tag{3.45}$$

In the simplest case of $N=3$, we have

$$P_\nu^{-1/2}(x)=\sqrt{\frac{2}{\pi}}(1-x^2)^{-1/4}\frac{\sin\left[\left(\nu+\frac{1}{2}\right)\arccos x\right]}{\nu+\frac{1}{2}}.\tag{3.46}$$

Using this representation of $P_\nu^{-1/2}(x)$ in equation (3.39), the latter being specialized to the case $N=3$, after some straightforward movements and with the help of the identity

$$\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)=\frac{\pi}{\cos(\pi\nu)},\tag{3.47}$$

we arrive at

$$\begin{aligned}G_\nu(\boldsymbol{r},\boldsymbol{r}')=&-\frac{1}{4\pi\cos(\pi\nu)}\frac{\sqrt{(r^2+\rho^2)(r'^2+\rho^2)}}{|\boldsymbol{r}-\boldsymbol{r}'|\sqrt{r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4}}\\&\times\sin\left[\left(\nu+\frac{1}{2}\right)\arccos\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)\right]\qquad(N=3).\end{aligned}\tag{3.48}$$

Equivalence between the result in equation (3.48) and the following expression (modified to conform with the present notation and corrected for a sign error)

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi\cos(\pi\nu)}\frac{\sqrt{(r^2+\rho^2)(r'^2+\rho^2)}}{|\boldsymbol{r}-\boldsymbol{r}'|\sqrt{r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4}} \times \sin\left[(2\nu+1)\arctan\frac{\sqrt{r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4}}{\rho|\boldsymbol{r}-\boldsymbol{r}'|}\right], \qquad (N=3), \tag{3.49}$$

given by Demkov and Ostrovsky in [4], may be easily established with the aid of the well-known inverse trigonometric identity

$$\arctan\xi = \frac{1}{2}\arccos\frac{1-\xi^2}{1+\xi^2} \qquad (\xi \geqslant 0). \tag{3.50}$$

4. The generalized Green's function for the fish-eye problem

A glance at either of equations (3.35), (3.39) or (3.40) reveals that the fish-eye Green's function $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ fails to exist for the following values of ν:

$$\nu = n + N/2 - 1 \quad \text{or} \quad \nu = -n - N/2 \qquad (n \in \mathbb{N}), \tag{4.1}$$

which, by the way, are the solutions of the quadratic equation

$$\nu(\nu+1) = \left(n+\frac{N}{2}\right)\left(n+\frac{N}{2}-1\right). \tag{4.2}$$

If either of the conditions set in equation (4.1) holds, one seeks the generalized Green's function $\bar{G}_{n+N/2-1}(\boldsymbol{r},\boldsymbol{r}') \equiv \bar{G}_{-n-N/2}(\boldsymbol{r},\boldsymbol{r}')$, defined through the limiting relation

$$\begin{aligned}\bar{G}_{n+N/2-1}(\boldsymbol{r},\boldsymbol{r}') &= \lim\nolimits_{\nu(\nu+1)\to(n+N/2)(n+N/2-1)} \\ &\quad\times \frac{\partial}{\partial[\nu(\nu+1)]}\left\{\left[\nu(\nu+1)-\left(n+\frac{N}{2}\right)\left(n+\frac{N}{2}-1\right)\right]G_\nu(\boldsymbol{r},\boldsymbol{r}')\right\} \\ &= \frac{1}{2n+N-1}\lim_{\nu\to n+N/2-1}\frac{\partial}{\partial\nu}\left[\left(\nu-n-\frac{N}{2}+1\right)\left(\nu+n+\frac{N}{2}\right)G_\nu(\boldsymbol{r},\boldsymbol{r}')\right].\end{aligned} \tag{4.3}$$

If, for instance, representation (3.39) of $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ is used in equation (4.3), this results in

$$\begin{aligned}\bar{G}_{n+N/2-1}(\boldsymbol{r},\boldsymbol{r}') &= (-)^n\frac{(n+N-2)!}{4\pi^{N/2}n!}\frac{\rho^{N/2-1}}{|\boldsymbol{r}-\boldsymbol{r}'|^{N/2-1}(r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4)^{N/4-1/2}} \\ &\quad\times\left\{\left[\psi(n+N-1)-\psi(n+1)+\frac{1}{2n+N-1}\right]\right. \\ &\quad\times P_{n+N/2-1}^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right) \\ &\quad\left.+\left.\frac{\partial P_\nu^{-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)}{\partial\nu}\right|_{\nu=n+N/2-1}\right\},\end{aligned} \tag{4.4}$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)}\frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta} \tag{4.5}$$

is the digamma function. A number of closed-form representations of the derivative $[\partial P_\nu^{-N/2+1}(x)/\partial\nu]_{\nu=n+N/2-1}$ required in equation (4.4) may be derived from the author's findings for $[\partial P_\nu^{\pm m}(z)/\partial\nu]_{\nu=n}$, $z \in \mathbb{C}\setminus(-1,1)$, presented in [13, 14]; the simplest, and thus potentially most useful, of these expressions are listed in the appendix.

In the particular case of $N = 3$, equation (4.4) yields simply

$$\begin{aligned}\bar{G}_{n+1/2}(\boldsymbol{r},\boldsymbol{r}') = {} & \frac{(-)^n}{4\pi^2}\frac{\sqrt{(r^2+\rho^2)(r'^2+\rho^2)}}{|\boldsymbol{r}-\boldsymbol{r}'|\sqrt{r^2r'^2+2\rho^2\boldsymbol{r}\cdot\boldsymbol{r}'+\rho^4}} \\ & \times\left\{\cos\left[(n+1)\arccos\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)\right]\right. \\ & \times\arccos\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right) \\ & \left.+\frac{\sin\left[(n+1)\arccos\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^2+\rho^2)}\right)\right]}{2(n+1)}\right\} \qquad (N=3). \end{aligned} \tag{4.6}$$

5. Prospective applications

The closed-form representations of the fish-eye Green's function $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ and of its generalized counterpart $\bar{G}_{n+N/2-1}(\boldsymbol{r},\boldsymbol{r}')$, found in this work, are certainly interesting for their own mathematical sake. They appear, however, to be also useful in the physical context. In a forthcoming report, we shall use them to show that, despite of claims to the contrary [4, 8, 9], in wave optics the infinite Maxwell fish-eye medium does not possess the same perfect focusing properties as it has in geometrical optics. Next, it has been confirmed [15] that the use of either of the closed-form expressions for $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ listed in section 3 simplifies greatly the mathematical analysis of wave-optical properties of cylindrical ($N = 2$) and spherical ($N = 3$) gradient-index lenses with the fish-eye refraction index (1.1) and of finite radii $r_{\text{lens}} \leqslant \rho\sqrt{2n_0-1}$. Finally, in yet another forthcoming paper, we shall show that there is a close mathematical relationship between the wavized Maxwell fish-eye problem in $\mathbb{R}^N$ and the N-dimensional Schrödinger–Coulomb problem in momentum space; in particular, we shall provide there an integral expression for the momentum-space Schrödinger–Coulomb Green's function in terms of the fish-eye Green's function discussed above.

Appendix. The derivatives $[\partial P_\nu^{-N/2+1}(x)/\partial\nu]_{\nu=n+N/2-1}$ for $N \in \mathbb{N}\setminus\{0,1\}$

From the relations

$$P_n^{-m}(x) = \mathrm{e}^{-\mathrm{i}\pi m/2}\frac{(n-m)!}{(n+m)!}P_n^m(x+\mathrm{i}0) \qquad (0 \leqslant m \leqslant n) \tag{A.1}$$

and

$$\left.\frac{\partial P_\nu^{-m}(x)}{\partial\nu}\right|_{\nu=n} = \mathrm{e}^{-\mathrm{i}\pi m/2}\frac{(n-m)!}{(n+m)!}\left.\frac{\partial P_n^m(x+\mathrm{i}0)}{\partial\nu}\right|_{\nu=n} - [\psi(n+m+1)-\psi(n-m+1)]P_n^{-m}(x) \qquad (0 \leqslant m \leqslant n), \tag{A.2}$$

and from a number of closed-form expressions for the derivative $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$, with $z \in \mathbb{C}\setminus(-1,1)$ and $0 \leqslant m \leqslant n$, found by the present author in [13, 14], one may derive, among others, the following representations of $[\partial P_\nu^{-N/2+1}(x)/\partial\nu]_{\nu=n+N/2-1}$, with $x \in [-1, 1]$ and with N *being an even natural number greater than zero*:[1]

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1} &= P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2} - 2\psi(n+N-1)P_{n+N/2-1}^{-N/2+1}(x)\\ &\quad + \frac{n!}{(n+N-2)!}\left(\frac{1-x^2}{4}\right)^{N/4-1/2}\sum_{k=0}^{n}(-)^k\frac{(k+n+N-2)!}{k!(k+N/2-1)!(n-k)!}\\ &\quad \times[2\psi(k+n+N-1)-\psi(k+N/2)]\left(\frac{1-x}{2}\right)^k + \left(\frac{1-x}{1+x}\right)^{N/4-1/2}\\ &\quad \times\sum_{k=0}^{n+N/2-1}(-)^k\frac{(k+n+N/2-1)!\psi(k+N/2)}{k!(k+N/2-1)!(n+N/2-k-1)!}\left(\frac{1-x}{2}\right)^k,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1} &= P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2} - 2\psi(n+N/2)P_{n+N/2-1}^{-N/2+1}(x) + \frac{n!}{(n+N-2)!}\\ &\quad \times\left(\frac{1-x^2}{4}\right)^{N/4-1/2}\sum_{k=0}^{n}(-)^k\frac{(k+n+N-2)!\psi(k+N/2)}{k!(k+N/2-1)!(n-k)!}\left(\frac{1-x}{2}\right)^k\\ &\quad + \left(\frac{1-x}{1+x}\right)^{N/4-1/2}\sum_{k=0}^{n+N/2-1}(-)^k\frac{(k+n+N/2-1)!}{k!(k+N/2-1)!(n+N/2-k-1)!}\\ &\quad \times[2\psi(k+n+N/2)-\psi(k+N/2)]\left(\frac{1-x}{2}\right)^k,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1} &= P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}\\ &\quad - [\psi(n+N-1)+\psi(n+N/2)]P_{n+N/2-1}^{-N/2+1}(x) + \frac{n!}{(n+N-2)!}\left(\frac{1-x^2}{4}\right)^{N/4-1/2}\\ &\quad \times\sum_{k=0}^{n}(-)^k\frac{(k+n+N-2)!\psi(k+n+N-1)}{k!(k+N/2-1)!(n-k)!}\left(\frac{1-x}{2}\right)^k + \left(\frac{1-x}{1+x}\right)^{N/4-1/2}\\ &\quad \times\sum_{k=0}^{n+N/2-1}(-)^k\frac{(k+n+N/2-1)!\psi(k+n+N/2)}{k!(k+N/2-1)!(n+N/2-k-1)!}\left(\frac{1-x}{2}\right)^k,\end{aligned} \tag{A.5}$$

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1} &= P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}\\ &\quad - (-)^n\left(\frac{1-x^2}{4}\right)^{-N/4+1/2}\sum_{k=0}^{N/2-2}\frac{(k+n)!(N/2-k-2)!}{k!(n+N-k-2)!}\left(\frac{1+x}{2}\right)^k\end{aligned}$$

---

[1] Attention! The *Mathematica* 7.0.0 function `LegendreP[nu,mu,2,x]` evaluates incorrectly (the modulus is correct but the sign is wrong) the numerical values of the associated Legendre functions $P_{2n+1}^{-2n-1}(x)$ and $P_{2n+2}^{-2n-1}(x)$ for $n \in \mathbb{N}$, $-1 \leqslant x \leqslant 1$. An empirically discovered remedy is to subject the variable x to the action of the function `SetPrecision` before it is used as an argument of `LegendreP`.

$$
\begin{aligned}
&+(-)^n\left(\frac{1+x}{1-x}\right)^{N/4-1/2}\sum_{k=0}^{n+N/2-1}(-)^k\frac{(k+n+N/2-1)!}{k!(k+N/2-1)!(n+N/2-k-1)!}\\
&\times[2\psi(k+n+N/2)-\psi(k+N/2)-\psi(k+1)]\left(\frac{1+x}{2}\right)^k,
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1}&=P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}-[\psi(n+N-1)-\psi(n+1)]P_{n+N/2-1}^{-N/2+1}(x)\\
&-(-)^n\frac{n!}{(n+N-2)!}\left(\frac{1-x}{1+x}\right)^{N/4-1/2}\\
&\times\sum_{k=0}^{N/2-2}\frac{(k+n+N/2-1)!(N/2-k-2)!}{k!(n+N/2-k-1)!}\left(\frac{1+x}{2}\right)^k\\
&+(-)^n\frac{n!}{(n+N-2)!}\left(\frac{1-x^2}{4}\right)^{N/4-1/2}\sum_{k=0}^{n}(-)^k\frac{(k+n+N-2)!}{k!(k+N/2-1)!(n-k)!}\\
&\times[2\psi(k+n+N-1)-\psi(k+N/2)-\psi(k+1)]\left(\frac{1+x}{2}\right)^k,
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1}&=P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}+[\psi(n+1)+\psi(n+N/2)]P_{n+N/2-1}^{-N/2+1}(x)\\
&-(-)^n n!(n+N/2-1)!\left(\frac{1+x}{1-x}\right)^{N/4-1/2}\left(\frac{1-x}{2}\right)^{n+N/2-1}\\
&\times\sum_{k=1}^{N/2-1}\frac{(k-1)!}{(k+n)!(k+n+N/2-1)!(N/2-k-1)!}\left(\frac{1-x}{1+x}\right)^k\\
&-n!(n+N/2-1)!\left(\frac{1-x}{1+x}\right)^{N/4-1/2}\left(\frac{1+x}{2}\right)^{n+N/2-1}\\
&\times\sum_{k=0}^{n}(-)^k\frac{\psi(n+N/2-k)+\psi(n-k+1)}{k!(k+N/2-1)!(n-k)!(n+N/2-k-1)!}\left(\frac{1-x}{1+x}\right)^k,
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial\nu}\right|_{\nu=n+N/2-1}&=P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}+[\psi(n+1)+\psi(n+N/2)]P_{n+N/2-1}^{-N/2+1}(x)\\
&-(-)^n n!(n+N/2-1)!\left(\frac{1-x}{1+x}\right)^{N/4-1/2}\left(\frac{1-x}{2}\right)^{n+N/2-1}\\
&\times\sum_{k=0}^{N/2-2}\frac{(N/2-k-2)!}{k!(n+N/2-k-1)!(n+N-k-2)!}\left(\frac{1+x}{1-x}\right)^k\\
&-(-)^n n!(n+N/2-1)!\left(\frac{1+x}{1-x}\right)^{N/4-1/2}\left(\frac{1-x}{2}\right)^{n+N/2-1}\\
&\times\sum_{k=0}^{n}(-)^k\frac{\psi(k+1)+\psi(k+N/2)}{k!(k+N/2-1)!(n-k)!(n+N/2-k-1)!}\left(\frac{1+x}{1-x}\right)^k,
\end{aligned}
\tag{A.9}
$$

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial \nu}\right|_{\nu=n+N/2-1} &= P_{n+N/2-1}^{-N/2+1}(x)\ln\frac{1+x}{2}\\ &\quad+[2\psi(2n+N-1)-\psi(n+N-1)-\psi(n+N/2)]P_{n+N/2-1}^{-N/2+1}(x)\\ &\quad+\sum_{k=0}^{n-1}(-)^{k+n}\frac{2k+N-1}{(n-k)(k+n+N-1)}\left[1+\frac{n!(k+N-2)!}{k!(n+N-2)!}\right]P_{k+N/2-1}^{-N/2+1}(x)\\ &\quad-\sum_{k=0}^{N/2-2}(-)^{k+n+N/2}\frac{2k+1}{(n+N/2-k-1)(k+n+N/2)}P_k^{-N/2+1}(x). \qquad \text{(A.10)}\end{aligned}$$

If, in turn, *N is an odd natural number greater than 1*, then from equations (3.44) and (3.45) one obtains

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial \nu}\right|_{\nu=n+N/2-1} &= \frac{2}{\pi}Q_{n+N/2-1}^{-N/2+1}(x)\arccos x\\ &\quad-\frac{\left(\frac{N-3}{2}\right)!}{2^{N/2-2}\sqrt{\pi}}\left(1-x^2\right)^{-N/4+1/2}\sum_{k=0}^{(N-3)/2}(-)^k\frac{(k+n)!}{k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!}\\ &\quad\times\left[\psi\left(k+n+\tfrac{N+1}{2}\right)-\psi(k+n+1)\right]\sin[(2k+n+1)\arccos x] \qquad \text{(A.11)}\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial P_\nu^{-N/2+1}(x)}{\partial \nu}\right|_{\nu=n+N/2-1} &= \frac{2}{\pi}Q_{n+N/2-1}^{-N/2+1}(x)\arccos x+\psi(n+1)P_{n+N/2-1}^{-N/2+1}(x)\\ &\quad-\sqrt{\frac{2}{\pi}}\,n!\sum_{k=0}^{(N-3)/2}\frac{\left(k+\frac{N-3}{2}\right)!\,\psi\left(k+n+\frac{N+1}{2}\right)}{2^k k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!}\\ &\quad\times\frac{\sin\left[\left(k+n+\frac{N-1}{2}\right)\arccos x+\left(k-\frac{N-3}{2}\right)\frac{\pi}{2}\right]}{(1-x^2)^{k/2+1/4}}, \qquad \text{(A.12)}\end{aligned}$$

where

$$\begin{aligned}P_{n+N/2-1}^{-N/2+1}(x) &= \frac{\left(\frac{N-3}{2}\right)!}{2^{N/2-2}\sqrt{\pi}}(1-x^2)^{-N/4+1/2}\sum_{k=0}^{(N-3)/2}(-)^k\frac{(k+n)!}{k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!}\\ &\quad\times\sin[(2k+n+1)\arccos x] \qquad \text{(A.13)}\end{aligned}$$

or, equivalently,

$$\begin{aligned}P_{n+N/2-1}^{-N/2+1}(x) &= \sqrt{\frac{2}{\pi}}\,n!\sum_{k=0}^{(N-3)/2}\frac{\left(k+\frac{N-3}{2}\right)!}{2^k k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!}\\ &\quad\times\frac{\sin\left[\left(k+n+\frac{N-1}{2}\right)\arccos x+\left(k-\frac{N-3}{2}\right)\frac{\pi}{2}\right]}{(1-x^2)^{k/2+1/4}} \qquad \text{(A.14)}\end{aligned}$$

and also

$$\begin{aligned}Q_{n+N/2-1}^{-N/2+1}(x) &= \frac{\sqrt{\pi}\left(\frac{N-3}{2}\right)!}{2^{N/2-1}}(1-x^2)^{-N/4+1/2}\sum_{k=0}^{(N-3)/2}(-)^k\frac{(k+n)!}{k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!}\\ &\quad\times\cos[(2k+n+1)\arccos x] \qquad \text{(A.15)}\end{aligned}$$

or, equivalently,

$$Q^{-N/2+1}_{n+N/2-1}(x) = \sqrt{\frac{\pi}{2}}\, n! \sum_{k=0}^{(N-3)/2} \frac{\left(k+\frac{N-3}{2}\right)!}{2^k k!\left(k+n+\frac{N-1}{2}\right)!\left(\frac{N-3}{2}-k\right)!} \times \frac{\cos\left[\left(k+n+\frac{N-1}{2}\right)\arccos x + \left(k-\frac{N-3}{2}\right)\frac{\pi}{2}\right]}{(1-x^2)^{k/2+1/4}}. \tag{A.16}$$

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