Sturmian basis functions for the harmonic oscillator

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We point out that the discrete Sturmian set for the one-dimensional harmonic oscillator, constructed recently by Antonsen [Phys. Rev. A **60**, 812 (1999)], is incomplete in $L^2_{x^2}(\mathbb{R})$ and thus does not form a basis in that Hilbert space. We show that for $E>0$, the spectrum of the Sturm-Liouville problem defining the Sturmian functions is mixed and consists of an infinite number of discrete positive eigenvalues, coinciding with those found by Antonsen, *and* the continuum of eigenvalues covering the negative real semiaxis. The discrete eigenvalues are simple while continuous eigenvalues are doubly degenerate. A basis in $L^2_{x^2}(\mathbb{R})$ comprises all Sturmian functions generated by that problem, i.e., those associated with discrete eigenvalues as well as those associated with continuum eigenvalues. We consider also the defining Sturm-Liouville problem in the case $E\leqslant 0$ and find that then the discrete part of its spectrum is absent: the spectrum is purely continuous, doubly degenerate, and covers the negative real semiaxis; the associated Sturmian functions form a noncountable basis in $L^2_{x^2}(\mathbb{R})$.

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I. INTRODUCTION

It is known that in many nonrelativistic quantum-mechanical problems involving the Coulomb potential, the so-called Schrödinger-Coulomb Sturmian functions are of particular value (e.g., [1–6]). These functions are generated by solving an eigenproblem consisting of the radial Schrödinger-Coulomb wave equation, with the *fixed* real energy E and the Coulomb potential strength chosen as an eigenvalue, augmented by usual quantum-mechanical boundary conditions. An extremely important feature of the Schrödinger-Coulomb Sturmian functions is that if E is fixed at some negative value, they form a discrete set, which, in contrast to the bound-state Schrödinger-Coulomb energy eigenfunctions, is complete and forms a basis in $L^2(\mathbb{R}_+)$.

The harmonic-oscillator problem is of comparable importance for nonrelativistic quantum mechanics as the Coulomb one. It is therefore natural to ask the following question: if one constructs the Sturmian functions for the harmonic oscillator, would the resulting eigenfunctions be equally useful in applications as their Coulombic counterparts are? An attempt to answer this question has been made recently by Antonsen [7] who claimed that for a fixed positive energy parameter E the one-dimensional harmonic-oscillator Sturmian functions form a discrete basis set suitable for practical use. Some intuitive premises, supported by our earlier experience in solving quantum-mechanical Sturm-Liouville problems [8–11], have caused us to doubt the validity of the conclusions of Ref. [7]. We have decided to reinvestigate the problem and the present paper reports the results of our study. We show that the functional set found by Antonsen is incomplete in $L^2_{x^2}(\mathbb{R})$: for $E>0$ the spectrum of the relevant Sturm-Liouville problem consists of an infinite number of discrete positive eigenvalues, coinciding with those found in Ref. [7], *and* the continuum of eigenvalues covering the negative real semiaxis, overlooked in Ref. [7]. The discrete eigenvalues are simple while the continuous eigenvalues are doubly degenerate. We consider also the case $E\leqslant 0$ and find that then the discrete part of the spectrum is absent; the spectrum is purely continuous, doubly degenerate, and covers the negative real semiaxis. The presence of the spectral continua prevents us from sharing the optimism of Ref. [7] about the suitability of the harmonic-oscillator Sturmian functions for use as a practical tool in solving actual quantum-mechanical problems.

II. STURMIAN BASIS FUNCTIONS FOR THE ONE-DIMENSIONAL HARMONIC OSCILLATOR

We define the Sturmian functions for the one-dimensional harmonic oscillator as solutions to the singular Sturm-Liouville problem composed of the differential equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\lambda\frac{m\omega^2x^2}{2}-E\right]S(x)=0 \quad (x\in\mathbb{R}) \tag{1}$$

augmented by the boundary conditions

$$S(x) \quad \text{bounded for} \quad x\to\pm\infty. \tag{2}$$

Here, $\omega>0$ is the oscillator's angular frequency, E is a real *fixed* parameter, while λ is an eigenvalue of the problem. The system (1) and (2) is Hermitian and the general theory of such systems [12–14] guarantees that all its eigenvalues are real and the corresponding eigenfunctions form a complete set.

To handle the differential equation (1), we introduce the dimensionless variable

$$\xi=\left(\frac{m\omega}{\hbar}\right)x^2 \quad (0\leqslant\xi<\infty) \tag{3}$$

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and seek $S(x)$ in the form

$$S(x)=\xi^{-1/4}F(\xi), \tag{4}$$

where $F(\xi)$ is to be determined. It is readily found from Eqs. (1), (3), and (4) that $F(\xi)$ is a solution to the differential equation

$$\left[\frac{d^2}{d\xi^2}+\frac{\frac{3}{16}}{\xi^2}+\frac{\frac{1}{2}\varepsilon}{\xi}-\frac{\lambda}{4}\right]F(\xi)=0, \tag{5}$$

where

$$\varepsilon=\frac{E}{\hbar\omega}. \tag{6}$$

The further progress requires considering the cases $E>0$ and $E\leqslant 0$ separately.

A. The case $E>0$

1. The case $\lambda>0$

In this case, Eq. (5) becomes

$$\left[\frac{d^2}{d(\beta\xi)^2}+\frac{\frac{3}{16}}{(\beta\xi)^2}+\frac{\eta}{(\beta\xi)}-\frac{1}{4}\right]F(\xi)=0, \tag{7}$$

where

$$\beta=\sqrt{\lambda}, \quad \eta=\frac{\varepsilon}{2\sqrt{\lambda}}>0. \tag{8}$$

Equation (7) is the Whittaker equation considered in the Appendix and its two linearly independent solutions are

$$F^{(e)}(\xi)=A^{(e)}M_{\eta,-1/4}(\beta\xi), \tag{9}$$

$$F^{(o)}(\xi)=A^{(o)}M_{\eta,1/4}(\beta\xi), \tag{10}$$

where $M_{\eta\gamma}(z)$ is the Whittaker function of the first kind and $A^{(e)}$ and $A^{(o)}$ are arbitrary nonzero constants [the meaning of the superscripts (e) and (o) will become clear shortly]. Hence, it follows that for $E>0$ and $\lambda>0$, two linearly independent solutions to Eq. (1) are

$$\tilde{S}^{(e)}(x)=B^{(e)}(\alpha x^2)^{-1/4}M_{\eta,-1/4}(\alpha x^2), \tag{11}$$

$$\tilde{S}^{(o)}(x)=B^{(o)}(\alpha x^2)^{-1/4}M_{\eta,1/4}(\alpha x^2), \tag{12}$$

where

$$\alpha=\left(\frac{m\omega}{\hbar}\right)\sqrt{\lambda}, \tag{13}$$

while $B^{(e)}$ and $B^{(o)}$ are nonzero constants. The tilde indicates that the functions (11) and (12) obey the differential equation (1) but not necessarily the boundary conditions (2). We notice that from the properties of the Whittaker function summarized in the Appendix, it follows that $\tilde{S}^{(e)}(x)$ and $\tilde{S}^{(o)}(x)$ are an even and an odd function of x, respectively. Therefore, investigating for what values of $\lambda>0$ the solutions (11) and (12) satisfy the asymptotic conditions (2), it suffices to consider the limit $x\to+\infty$.

It is known [cf. Eq. (A4)] that for large positive real arguments, the Whittaker function $M_{\eta\gamma}(z)$ diverges exponentially unless its indices are related by $\eta-\gamma-\frac{1}{2}=n$, where $n=0,1,2,\ldots$ This implies that the even solutions (11) are asymptotically bounded if

$$\lambda_n^{(e)}=\frac{4\varepsilon^2}{(4n+1)^2} \quad (n=0,1,2,\ldots), \tag{14}$$

while the odd solutions (12) are bounded if

$$\lambda_n^{(o)}=\frac{4\varepsilon^2}{(4n+3)^2} \quad (n=0,1,2,\ldots). \tag{15}$$

Equations (14) and (15) may be combined, yielding the discrete nondegenerate part of the spectrum of the problem (1) and (2) for $E>0$:

$$\lambda_n=\frac{4\varepsilon^2}{(2n+1)^2} \quad (n=0,1,2,\ldots). \tag{16}$$

It stems from relations (A7) and (A8) of the Appendix that the corresponding unnormalized eigenfunctions are

$$S_n(x)=C_nH_n(kx/\sqrt{2n+1})e^{-k^2x^2/2(2n+1)} \quad (n=0,1,2,\ldots), \tag{17}$$

where $H_n(z)$ is the Hermite polynomial and

$$k=\sqrt{2mE/\hbar^2}. \tag{18}$$

The eigenfunctions labeled by even or odd indices n are even or odd functions of x, respectively. It is then a simple exercise to show that eigenfunctions $S_n(x)$ and $S_{n'}(x)$ corresponding to different discrete eigenvalues λ_n and $\lambda_{n'}$ are orthogonal in the sense of

$$\int_{-\infty}^{\infty}dx\,x^2S_n(x)S_{n'}(x)=0 \quad (n\neq n'), \tag{19}$$

and that, if the factor C_n in Eq. (17) is chosen so that

$$S_n(x)=\frac{k^{3/2}}{\pi^{1/4}2^{(n-1)/2}(n!)^{1/2}(2n+1)^{5/4}}\times H_n(kx/\sqrt{2n+1})e^{-k^2x^2/2(2n+1)}, \tag{20}$$

Eq. (19) is generalized to the orthonormality relation

$$\int_{-\infty}^{\infty}dx\,x^2S_n(x)S_{n'}(x)=\delta_{nn'}. \tag{21}$$

So far our results agree with those obtained in Ref. [7]. However, we have not considered yet the possibility of existence of negative eigenvalues λ.

2. The case λ<0

In this case, Eq. (5) may be rewritten in the form

$$\left[\frac{d^2}{d(i\beta\xi)^2}+\frac{\frac{3}{16}}{(i\beta\xi)^2}+\frac{-i\eta}{(i\beta\xi)}-\frac{1}{4}\right]F(\xi)=0, \tag{22}$$

where now

$$\beta=\sqrt{-\lambda}, \quad \eta=\frac{\varepsilon}{2\sqrt{-\lambda}}>0. \tag{23}$$

Equation (10) is again the Whittaker equation and its two linearly independent solutions are

$$F^{(e)}(\xi)=A^{(e)}M_{-i\eta,-1/4}(i\beta\xi), \tag{24}$$

$$F^{(o)}(\xi)=A^{(o)}M_{-i\eta,1/4}(i\beta\xi), \tag{25}$$

where $A^{(e)}$ and $A^{(o)}$ are nonzero constants. The corresponding linearly independent solutions to Eq. (1) for $E>0$ and $\lambda<0$ are

$$\tilde{S}^{(e)}(x)=B^{(e)}(i\alpha x^2)^{-1/4}M_{-i\eta,-1/4}(i\alpha x^2) \tag{26}$$

and

$$\tilde{S}^{(o)}(x)=B^{(o)}(i\alpha x^2)^{-1/4}M_{-i\eta,1/4}(i\alpha x^2), \tag{27}$$

where this time

$$\alpha=\left(\frac{m\omega}{\hbar}\right)\sqrt{-\lambda} \tag{28}$$

and $B^{(e)}$ and $B^{(o)}$ are nonzero constants. Again, the tilde indicates that so far we have not taken care of the boundary conditions (2). The functions $\tilde{S}^{(e)}(x)$ and $\tilde{S}^{(o)}(x)$ are an even and an odd function of the variable x, respectively.

We now ask the following question: for what values of $\lambda<0$ are the functions $\tilde{S}^{(e)}(x)$ or $\tilde{S}^{(o)}(x)$ bounded for $x\to+\infty$? [The limit $x\to-\infty$ need not be considered separately due to the aforementioned symmetry of $\tilde{S}^{(e)}(x)$ and $\tilde{S}^{(o)}(x)$.] To answer the question posed, we make use of the asymptotic expansion (A4) of the Whittaker function, obtaining

$$\tilde{S}^{(e)}(x) \underset{x\to\infty}{\sim} B^{(e)}\frac{2\sqrt{\pi}e^{i\pi/8-\pi\eta/2}}{|\Gamma(\frac{1}{4}+i\eta)|}(i\alpha x^2)^{-1/4} \times\cos[\tfrac{1}{2}\alpha x^2+\eta\ln(\alpha x^2)-\tfrac{1}{8}\pi-\sigma^{(e)}(\eta)] \tag{29}$$

and

$$\tilde{S}^{(o)}(x) \underset{x\to\infty}{\sim} B^{(o)}\frac{\sqrt{\pi}e^{i3\pi/8-\pi\eta/2}}{|\Gamma(\frac{3}{4}+i\eta)|}(i\alpha x^2)^{-1/4} \times\cos[\tfrac{1}{2}\alpha x^2+\eta\ln(\alpha x^2)-\tfrac{3}{8}\pi-\sigma^{(o)}(\eta)], \tag{30}$$

where

$$\sigma^{(e)}(\eta)=\arg\Gamma(\tfrac{1}{4}+i\eta), \quad \sigma^{(o)}(\eta)=\arg\Gamma(\tfrac{3}{4}+i\eta). \tag{31}$$

From Eqs. (29) and (30) we infer that for an *arbitrary* $\lambda<0$ *both* functions $\tilde{S}^{(e)}(x)$ and $\tilde{S}^{(o)}(x)$ vanish for $x\to+\infty$ and therefore obey the boundary conditions (2). This implies that the spectrum of the Sturm-Liouville problem (1) and (2) is mixed and apart from the infinite set of the discrete positive eigenvalues given by Eq. (16), the problem possesses also the continuous eigenvalues covering the negative real semiaxis. This fact has been overlooked in Ref. [7]. In contrast to the case of the discrete eigenvalues, which, as we have shown, are simple, the continuous eigenvalues are doubly degenerate. Henceforth, the tilde over the continuum eigenfunctions will be omitted.

It remains to investigate the problem of the orthogonality and normalization of the continuum eigenfunctions. We shall discuss in details the case of the even eigenfunctions $S_{\lambda}^{(e)}(x)$ and provide only final results for the odd eigenfunctions $S_{\lambda}^{(o)}(x)$ because both cases are analogous. Consider thus two differential equations of the form (1) obeyed by two different even eigenfunctions $S_{\lambda}^{(e)}(x)$ and $S_{\lambda'}^{(e)}(x)$. We premultiply the equation for $S_{\lambda}^{(e)}(x)$ by $S_{\lambda'}^{(e)}(x)$, the equation for $S_{\lambda'}^{(e)}(x)$ by $S_{\lambda}^{(e)}(x)$, subtract, and integrate the result over x from $x=-X$ to $x=+X$, where $X>0$. On employing the Green's integration theorem, this yields

$$(\lambda-\lambda')\left(\frac{m\omega}{\hbar}\right)^2\int_{-X}^{+X}dx\,x^2S_{\lambda}^{(e)}(x)S_{\lambda'}^{(e)}(x) =\left[S_{\lambda'}^{(e)}(x)\frac{dS_{\lambda}^{(e)}(x)}{dx}-S_{\lambda}^{(e)}(x)\frac{dS_{\lambda'}^{(e)}(x)}{dx}\right]_{-X}^{+X}, \tag{32}$$

which, after making use of the evenness property of the eigenfunctions considered (their first derivatives are odd), may be rewritten in the form

$$\int_{-X}^{+X}dx\,x^2S_{\lambda}^{(e)}(x)S_{\lambda'}^{(e)}(x)=\frac{2(\hbar/m\omega)^2}{\lambda-\lambda'}\left[S_{\lambda'}^{(e)}(x)\frac{dS_{\lambda}^{(e)}(x)}{dx} -S_{\lambda}^{(e)}(x)\frac{dS_{\lambda'}^{(e)}(x)}{dx}\right]_{x=X}. \tag{33}$$

For large X, the functions on the right-hand side of the above equation may be replaced with their asymptotic forms (29). This yields, after simple trigonometric manipulations,

$$\int_{-X}^{+X} dx\, x^2 S_\lambda^{(e)}(x) S_{\lambda'}^{(e)}(x) \underset{X\to\infty}{\sim} -B_\lambda^{(e)} B_{\lambda'}^{(e)} \left(\frac{\hbar}{m\omega}\right)^2 \frac{4\pi e^{-\pi(\eta+\eta')/2}(\alpha\alpha')^{-1/4}}{|\Gamma(\frac{1}{4}+i\eta)\Gamma(\frac{1}{4}+i\eta')|}\left(\frac{\alpha^2-\alpha'^2}{\lambda-\lambda'}\right)$$
$$\times\left[\frac{\sin[\frac{1}{2}(\alpha+\alpha')X^2+\eta\ln(\alpha X^2)+\eta'\ln(\alpha' X^2)-\frac{1}{4}\pi-\sigma^{(e)}(\eta)-\sigma^{(e)}(\eta')]}{\alpha+\alpha'} + \frac{\sin[\frac{1}{2}(\alpha-\alpha')X^2+\eta\ln(\alpha X^2)-\eta'\ln(\alpha' X^2)-\sigma^{(e)}(\eta)+\sigma^{(e)}(\eta')]}{\alpha-\alpha'}\right]. \tag{34}$$

In the limit $X\to\infty$, on utilizing the following well-known representation of the Dirac delta function

$$\delta(t-t') = \lim_{a\to\infty} \frac{\sin[a(t-t')f(a,t,t')]}{\pi(t-t')} \quad [\lim_{a\to\infty} f(a,t,t')=1], \tag{35}$$

we obtain

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(e)}(x) S_{\lambda'}^{(e)}(x) = B_\lambda^{(e)} B_{\lambda'}^{(e)} \frac{4\pi^2 e^{-\pi(\eta+\eta')/2}(\alpha\alpha')^{-1/4}}{|\Gamma(\frac{1}{4}+i\eta)\Gamma(\frac{1}{4}+i\eta')|} \times[\delta(\alpha+\alpha')+\delta(\alpha-\alpha')]. \tag{36}$$

Since both α and α' are positive [cf. Eq. (28)], the argument of $\delta(\alpha+\alpha')$ does not vanish; consequently, the first delta is effectively zero and may be omitted in further considerations. The second delta may be transformed by utilizing the identity [cf. again Eq. (28)]

$$\delta(\alpha-\alpha') = 2\left(\frac{\hbar}{m\omega}\right)^2 \alpha\,\delta(\lambda-\lambda'), \tag{37}$$

which yields

$$\int_{-\infty}^{+\infty} dx\, x^2 S_\lambda^{(e)}(x) S_{\lambda'}^{(e)}(x) = (B_\lambda^{(e)})^2 \left(\frac{\hbar}{m\omega}\right)^2 \frac{8\pi^2 e^{-\pi\eta}\alpha^{1/2}}{|\Gamma(\frac{1}{4}+i\eta)|^2} \times\delta(\lambda-\lambda'). \tag{38}$$

Equation (38) implies that the even continuum eigenfunctions

$$S_\lambda^{(e)}(x) = \left(\frac{m\omega}{\hbar}\right) \times \frac{e^{\pi\eta/2}|\Gamma(\frac{1}{4}+i\eta)|}{2^{3/2}\pi\alpha^{1/4}} (i\alpha x^2)^{-1/4} M_{-i\eta,-1/4}(i\alpha x^2) \tag{39}$$

are orthonormal in the sense of

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(e)}(x) S_{\lambda'}^{(e)}(x) = \delta(\lambda-\lambda'). \tag{40}$$

In the analogous way one shows that the odd continuum eigenfunctions normalized and orthogonal according to

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(o)}(x) S_{\lambda'}^{(o)}(x) = \delta(\lambda-\lambda') \tag{41}$$

are

$$S_\lambda^{(o)}(x) = \left(\frac{m\omega}{\hbar}\right) \times \frac{e^{\pi\eta/2-i\pi/4}|\Gamma(\frac{3}{4}+i\eta)|}{2^{1/2}\pi\alpha^{1/4}} (i\alpha x^2)^{-1/4} M_{-i\eta,1/4}(i\alpha x^2). \tag{42}$$

It is easy to verify by applying the complex conjugation formula (A6) that the functions (39) and (42) are real. Moreover, we have an obvious orthogonality relation

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(o)}(x) S_{\lambda'}^{(e)}(x) = 0. \tag{43}$$

Finally, in the same way in which one proves the relation (19), it is found that the continuum eigenfunctions are orthogonal to those belonging to the discrete spectrum in the sense of

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(e)}(x) S_n(x) = 0 \tag{44}$$

and

$$\int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(o)}(x) S_n(x) = 0. \tag{45}$$

The set of eigenfunctions $\{S_n(x)\}\cup\{S_\lambda^{(e)}(x)\cup\{S_\lambda^{(o)}(x)\}$ is complete in $L^2_{x^2}(\mathbb{R})$ and the following closure relation holds [15]:

$$\sum_{n=0}^{\infty} S_n(x)S_n(x') + \int_{-\infty}^{0} d\lambda\, S_\lambda^{(e)}(x)S_\lambda^{(e)}(x') + \int_{-\infty}^{0} d\lambda S_\lambda^{(o)}(x)S_\lambda^{(o)}(x') = \frac{\delta(x-x')}{xx'} \quad (x,x' \in \mathbb{R}). \tag{46}$$

Consequently, an arbitrary function $F(x) \in L^2_{x^2}(\mathbb{R})$ has an expansion

$$F(x) = \sum_{n=0}^{\infty} c_n S_n(x) + \int_{-\infty}^{0} d\lambda\, c_\lambda^{(e)} S_\lambda^{(e)}(x) + \int_{-\infty}^{0} d\lambda\, c_\lambda^{(o)} S_\lambda^{(o)}(x) \quad (x \in \mathbb{R}), \tag{47}$$

where

$$c_n = \int_{-\infty}^{\infty} dx\, x^2 S_n(x) F(x), \tag{48}$$

$$c_\lambda^{(e)} = \int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(e)}(x) F(x), \tag{49}$$

$$c_\lambda^{(o)} = \int_{-\infty}^{\infty} dx\, x^2 S_\lambda^{(o)}(x) F(x). \tag{50}$$

Notice that in general $c_\lambda^{(e)} \neq 0$, $c_\lambda^{(o)} \neq 0$, and therefore

$$F(x) \neq \sum_{n=0}^{\infty} c_n S_n(x). \tag{51}$$

B. The case $E \leqslant 0$

The examination of this case proceeds in the way analogous to that of the preceding subsection. One finds, however, that this time the discrete part of the spectrum is absent: the λ spectrum is purely continuous, doubly degenerate, and coincides with the real negative semiaxis. Even and odd continuum eigenfunctions corresponding to some particular eigenvalue λ, chosen to be real and orthonormal in the sense of Eqs. (40) and (41), are

$$S_\lambda^{(e)}(x) = \left(\frac{m\omega}{\hbar}\right) \times \frac{e^{\pi\eta/2}|\Gamma(\frac{1}{4}+i\eta)|}{2^{3/2}\pi\alpha^{1/4}} (i\alpha x^2)^{-1/4} M_{-i\eta,-1/4}(i\alpha x^2) \tag{52}$$

and

$$S_\lambda^{(o)}(x) = \left(\frac{m\omega}{\hbar}\right) \times \frac{e^{\pi\eta/2 - i\pi/4}|\Gamma(\frac{3}{4}+i\eta)|}{2^{1/2}\pi\alpha^{1/4}} (i\alpha x^2)^{-1/4} M_{-i\eta,1/4}(i\alpha x^2), \tag{53}$$

respectively. The eigenfunctions (52) and (53) differ from seemingly identical eigenfunctions (39) and (42), since now

$$\eta = \frac{\varepsilon}{2\sqrt{-\lambda}} \leqslant 0 \tag{54}$$

[cf. Eq. (23)]. The parameter α is defined as in Eq. (28). In the limiting case $E=0$ the eigenfunctions (52) and (53) may be expressed in terms of the Bessel functions. On utilizing the relation (A9), one arrives at

$$S_\lambda^{(e)}(x) = \left(\frac{m\omega}{\hbar}\right) 2^{-3/2} x^{1/2} J_{-1/4}(\tfrac{1}{2}\alpha x^2) \quad (E=0), \tag{55}$$

$$S_\lambda^{(o)}(x) = \left(\frac{m\omega}{\hbar}\right) 2^{-3/2} x^{1/2} J_{1/4}(\tfrac{1}{2}\alpha x^2) \quad (E=0). \tag{56}$$

The set consisting of the eigenfunctions (52) and (53) is complete in $L^2_{x^2}(\mathbb{R})$ [15]. The corresponding closure relation is [cf. Eq. (46)]

$$\int_{-\infty}^{0} d\lambda\, S_\lambda^{(e)}(x)S_\lambda^{(e)}(x') + \int_{-\infty}^{0} d\lambda\, S_\lambda^{(o)}(x)S_\lambda^{(o)}(x') = \frac{\delta(x-x')}{xx'} \quad (x,x' \in \mathbb{R}). \tag{57}$$

An arbitrary function $F(x) \in L^2_{x^2}(\mathbb{R})$ possesses the expansion

$$F(x) = \int_{-\infty}^{0} d\lambda\, c_\lambda^{(e)} S_\lambda^{(e)}(x) + \int_{-\infty}^{0} d\lambda\, c_\lambda^{(o)} S_\lambda^{(o)}(x) \quad (x \in \mathbb{R}) \tag{58}$$

with the coefficients $c_\lambda^{(e)}$ and $c_\lambda^{(o)}$ given by formulas analogous to Eqs. (49) and (50).

III. CONCLUSIONS

In response to the recent work [7], in this paper we have reconsidered the problem of constructing the Sturmian basis functions for the harmonic oscillator. We have obtained the most important results in the case when the energy parameter entering the defining Sturm-Liouville problem is positive. We have shown that then the spectrum of potential strengths is *mixed* and contains the discrete nondegenerate positive eigenvalues, given by Eq. (16) and coinciding with those found in Ref. [7], *and* the continuum of doubly degenerate negative eigenvalues, overlooked in Ref. [7]. The presence of the continuous part of the spectrum causes that eigenfunctions corresponding to discrete eigenvalues do *not* form a complete set in $L^2_{x^2}(\mathbb{R})$ (i.e., do *not* span that Hilbert space) and, consequently, in general *cannot* be used as an expansion basis. A complete basis set in $L^2_{x^2}(\mathbb{R})$ is formed by the larger set comprising the discrete *and* the continuous Sturmian functions but, because of the presence of the latter, we are skeptical about the suitability of this basis for use as a tool for solving actual quantum-mechanical problems.

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APPENDIX: CERTAIN PROPERTIES OF THE WHITTAKER FUNCTION

In this appendix we summarize those properties of the Whittaker function of the first kind, $M_{\eta\gamma}(z)$, which have been useful in studying properties of the harmonic oscillator Sturmian functions. A more comprehensive treatment of the Whittaker function may be found in Refs. [16,17].

The function $M_{\eta\gamma}(z)$ is defined in terms of the confluent hypergeometric function

$$ {}_1F_1(a;b;z)=1+\frac{a}{b}\,\frac{z}{1!}+\frac{a(a+1)}{b(b+1)}\,\frac{z^2}{2!}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\,\frac{z^3}{3!}+\cdots \tag{A1} $$

in the following way:

$$ M_{\eta\gamma}(z)=z^{\gamma+1/2}e^{-z/2}\,{}_1F_1(\gamma+\tfrac{1}{2}-\eta;2\gamma+1;z). \tag{A2} $$

It is a solution of the Whittaker differential equation

$$ \left[\frac{d^2}{dz^2}-\frac{\gamma^2-\frac{1}{4}}{z^2}+\frac{\eta}{z}-\frac{1}{4}\right]M_{\eta\gamma}(z)=0. \tag{A3} $$

For large values of the argument the function $M_{\eta\gamma}(z)$ has the asymptotic expansion

$$ M_{\eta\gamma}(z)\xrightarrow{|z|\to\infty} z^{\gamma+1/2}e^{-z/2}\left[\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+\frac{1}{2}+\eta)}(-z)^{-\gamma-1/2+\eta}+\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+\frac{1}{2}-\eta)}z^{-\gamma-1/2-\eta}e^{z}\right], \tag{A4} $$

where

$$ -z=e^{i\pi\Delta(z)}z,\quad \Delta(z)=\begin{cases}-1 & \text{for } 0<\arg z<\pi\\ +1 & \text{for } -\pi<\arg z<0.\end{cases} \tag{A5} $$

The following complex conjugation formula holds:

$$ [M_{\alpha+i\beta,\gamma}(ix)]^{*}=e^{-i\pi(\gamma+1/2)}M_{-\alpha+i\beta,\gamma}(ix) \quad (\alpha,\beta,\gamma\in\mathbb{R},x\in\mathbb{R}_+). \tag{A6} $$

A variety of special functions may be expressed in terms of the Whittaker function of the first kind. In this work we have used the relations

$$ M_{n+1/4,-1/4}(z^2)=(-)^n\frac{n!}{(2n)!}z^{1/2}e^{-z^2/2}H_{2n}(z), \tag{A7} $$

$$ M_{n+3/4,1/4}(z^2)=(-)^n\frac{n!}{2(2n+1)!}z^{1/2}e^{-z^2/2}H_{2n+1}(z), \tag{A8} $$

where $H_n(z)$ is the Hermite polynomial, and

$$ M_{0,\gamma}(iz)=2^{2\gamma}\Gamma(\gamma+1)e^{i\pi(2\gamma+1)/4}z^{1/2}J_{\gamma}(\tfrac{1}{2}z), \tag{A9} $$

where $J_{\gamma}(z)$ is the Bessel function.

[1] M. Rotenberg, Adv. At. Mol. Phys. **6**, 233 (1970).
[2] E. J. Weniger, J. Math. Phys. **26**, 276 (1985).
[3] N. L. Manakov, V. D. Ovsiannikov, and L. P. Rapoport, Phys. Rep. **141**, 319 (1986).
[4] E. Karule and R. H. Pratt, J. Phys. B **24**, 1585 (1991).
[5] N. L. Manakov, A. Maquet, S. I. Marmo, and C. Szymanowski, Phys. Lett. A **237**, 234 (1998).
[6] A. Maquet, V. Véniard, and T. A. Marian, J. Phys. B **31**, 3743 (1998).
[7] F. Antonsen, Phys. Rev. A **60**, 812 (1999).
[8] R. Szmytkowski, J. Phys. B **30**, 825 (1997); **30**, 2747(E) (1997).
[9] R. Szmytkowski, J. Phys. A **31**, 4963 (1998); **31**, 7415(E) (1998).
[10] R. Szmytkowski, e-print physics/9902050.
[11] R. Szmytkowski, J. Phys. A **33**, 427 (2000).
[12] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations Pts. I & II* (Oxford University Press, Oxford, 1958–1962).
[13] B. M. Levitan and I. S. Sargsjan, *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators* (American Mathematical Society, Providence, RI, 1975).
[14] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators* (Nauka, Moscow, 1988) (in Russian).
[15] The validity of the relations (46) and (57) follows from general theorems of functional analysis [12–14] but may be also proved directly in the way analogous to that in which N. Mukunda [Am. J. Phys. **46**, 910 (1978)] proved the completeness of the radial Coulomb wave functions.
[16] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer, Berlin, 1966).
[17] H. Buchholz, *The Confluent Hypergeometric Function* (Springer, Berlin, 1969).