Fourier transforms on Cantor sets: A study in non-Diophantine arithmetic and calculus

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\textbf{ABSTRACT}

Fractals equipped with intrinsic arithmetic lead to a natural definition of differentiation, integration, and complex structure. Applying the formalism to the problem of a Fourier transform on fractals we show that the resulting transform has all the required basic properties. As an example we discuss a sawtooth signal on the ternary middle-third Cantor set. The formalism works also for fractals that are not self-similar.

Four different definitions of a gradient (due to Kusuoka, Kigami, Strichartz and Teplyaev) can be found in [10].

One might naively expect that it would be more logical to begin with first derivatives and only then turn to higher-order operators, such as Laplacians. It turns out that Laplacians defined in the above ways cannot be regarded as second-order operators. Still, an approach where Laplacians are indeed second-order is possible and was introduced by Fujita [11,12], and further developed by Freiberg, Zähle and others [13–17]. We will later see that a non-Diophantine Laplacian is exactly second-order and, similarly to the approach from [13–17], is based on derivatives and integrals satisfying the fundamental laws of calculus.

In yet another traditional approach to harmonic analysis on fractals, one begins with self-similar fractal measures, and then seeks exponential functions that are orthogonal and complete with respect to them. The classic result of Jorgensen and Pedersen [18] states that such exponential functions do exist on certain fractals, such as the quaternary Cantor set, but cannot be constructed in the important case of the ternary middle-third Cantor set.

In the present paper, we will follow a different approach. One begins with arithmetic operations (addition, subtraction, multiplication, and division) which are intrinsic to the fractal. The arithmetic so defined is non-Diophantine in the sense of Burgin [19,20]. An important step is then to switch from arithmetic to calculus [21] where, in particular, derivatives and integrals are naturally defined. The resulting formalism is simple and general, extends beyond fractal applications, but works with no difficulty for Cantorian fractals, even if they are not self-similar [21,22]. Actually, a straightforward motivation for the present paper came from discussions with the referee of [22], who pointed out

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possible difficulties with momentum representation of quantum mechanics on Cantorian space-times. In the sequel to the present paper [27], we show how to generalize the construction to fractals of a Sierpiński type.

In Section 2 we recall the basic properties of non-Diophantine arithmetic, illustrated by four examples from physics, cognitive science, and fractal theory. Section 3 is devoted to complex numbers, discussed along the lines proposed by one of us in [21], and with particular emphasis on trigonometric and exponential functions. In Section 4 we recall the non-Diophantine-arithmetic definitions of derivatives and integrals. Section 5 discusses a scalar product of functions, and the corresponding Fourier transform (both complex and real) is introduced in Section 7. In Section 8 we discuss an explicit example of a sawtooth signal with Cantorian domain and range. Finally, in Section 9 we briefly discuss the issue of spectrum of Fourier frequencies, and compare our results with those from [18].

2. Generalized arithmetic: Fractal and not only

Consider a set $X$ and a bijection $f: X \rightarrow \mathbb{R}$ following the general formalism from [21] we define the arithmetic operations in $X$:

$x \otimes y = f^{-1}(f(x) + f(y))$, 

$x \odot y = f^{-1}(f(x) - f(y))$, 

$x \odot y = f^{-1}(f(xy)f(y))$, 

$x \odot y = f^{-1}(f(x)/f(y))$, 

for any $x, y \in X$. In later applications we will basically concentrate on an appropriately constructed fractal $X$, but the results are more general. This is an example of a non-Diophantine arithmetic [19,20].

One verifies the standard properties: (1) associativity $(x \otimes y) \oplus z = x \otimes (y \oplus z)$, $(x \otimes y) \oplus z = x \otimes (y \oplus z)$, (2) commutativity $x \odot y = y \odot x$, $x \odot y = y \odot x$, (3) distributivity $(x \otimes y) \oplus z = (x \otimes z) \oplus (y \otimes z)$. Elements $0', 1' \in X$ are defined by $0' \oplus x = x$, $1' \odot x = x$, which implies $f(0') = 0$, $f(1') = 1$. One further finds $x \odot x = 0'$, $x \otimes x = 1'$, as expected. A negative of $x \in X$ is defined as $\ominus x = 0' \oplus x = f^{-1}(-f(x))$, i.e. $f(\ominus x) = -f(x)$ and $f(x \odot y) = f(1') = -1$, i.e. $\ominus 1' = f^{-1}(-1)$. Notice that

$\ominus 1' = f^{-1}(f(1')^2) = f^{-1}(1') = 1'$. (1)

Multiplication can be regarded as repeated addition in the following sense. Let $n \in \mathbb{N}$ and $n' = f^{-1}(n) \in \mathbb{N}$. Then

$n' \oplus m' = (n + m)'$, 

$n' \odot m' = (nm)'$, 

$n' \oplus m' = \underbrace{(m' \oplus \cdots \oplus m')}_{\text{n times}}'. 

In particular $n' = 1' \oplus \cdots \oplus 1'$ (n times).

A power function $A(x) = x \odot \cdots \odot x$ (n times) will be denoted by $x^n$. Such a notation is consistent in the sense that $x^n \odot y^m = x^{n+m} = x^n \oplus y^m$. (5)

Before we plunge into fractal applications let us consider four explicit examples of non-Diophantine arithmetic.

2.1. Benoît's number scaling

The number-scaling approach of Benoît [23,24] can be regarded as a particular case of the above formalism with $f(x) = px$, $p \neq 0$. Indeed, $x \odot y = (1/p)(pxy) = px$, $x \odot y = (1/p)(pxy) = x + y$, $x \odot y = (1/p)(pxy)/(py) = x/(py)$, but $f(1/p) = 1$. Since $(1/p) \odot x = (1/p)(p(1/p)px) = x$ one infers that $1' = f^{-1}(1) = 1/p$ is the unit element of multiplication in Benoît's non-Diophantine arithmetic.

2.2. Fechner map

This arithmetic is implicitly used in cognitive science [25]. It occurs as a solution of the following Weber–Fechner problem [26]: Find a generalized arithmetic such that $(x + kn) \oplus x$ is independent of $x$. Here $x \rightarrow x' = x + \Delta x$ is the change of an input signal, while $x' \odot x$ is the change of $x$ as perceived by a nervous system. Experiments show that $\Delta x/x \approx k = \text{const}$ (Weber–Fechner law) in a wide range of $x$, and with different values of $k$ for different types of stimuli. The corresponding arithmetic is defined by the ‘Fechner map’ $f(x) = a \ln x + b$, $f^{-1}(x) = e^{(x-b)/a}$, and thus $0' = f^{-1}(0) = e^{-b/a}, 1' = f^{-1}(1) = e^{(1-b)/a}$. Clearly, $0' \neq 0$ and $1' \neq 1$. Interestingly, the Fechnerian negative of $x \in \mathbb{R}$, reads

$x \ominus x = 0' \oplus x = e^{b/a}x \in \mathbb{R}$. 

but nevertheless does satisfy

$x \ominus x = e^{-b/a} = 0'$, (7)

as it should on general grounds [25]. So, numbers that are negative with respect to one arithmetic are positive with respect to another. In a future work we will show that Fechner’s $f$ has intriguing consequences for relativistic physics.

2.3. Ternary Cantor line

Let us start with the right-open interval $[0,1) \subset \mathbb{R}$, and let the (countable) set $\mathbb{Y}_2 \subset [0,1)$ consist of those numbers that have two different binary representations. Denote by $0.t_1t_2\ldots$ a ternary representation of some $x \in [0,1)$. If $y \in \mathbb{Y}_2 = \{0.1\} \setminus \mathbb{Y}_2$ then $y$ has a unique binary representation, say $y = 0.b_1b_2\ldots$. Then one sets $g_2(y) = 0.t_1t_2\ldots t_j = 2b_j$. The index $j$ appears for the following reason. Let $y = 0.b_1b_2\ldots b_jb_j'\ldots$, $z = 0.b_jb_1'\ldots$ be the two representations of $y \in \mathbb{Y}_2$. There are two options, so we define: $g_2(y) = \min\{0.t_1t_2\ldots, 0.t_jt_j'\ldots, \ldots\}$, and $g_1(y) = \max\{0.t_1t_2\ldots, 0.t_jt_j'\ldots, \ldots\}$, then $t_j = 2b_j$, $t_j' = 2b_j'$. We have therefore constructed two injective maps $g_2: [0,1) \rightarrow [0,1)$, and the ternary Cantor-like sets are defined as the images $C_1(0,1) = g_2(\mathbb{Y}_2)$, and $C_2 = C_1(0,1) \setminus \{0.1\}$, $f_2 = g_2^{-1}$, is a bijection between $C_1(0,1)$ and the interval. For example, $1/2 \in \mathbb{Y}_2$ since $1/2 = 0.1_2 = 0.01_2$. We find

$g_2(1/2) = \min\{0.2_3 = 2/3, 0.0(2)_3 = 1/3\} = 1/3$, 

$g_1(1/2) = \max\{0.2_3 = 2/3, 0.0(2)_3 = 1/3\} = 2/3$. 

Accordingly, $1/3 \in C_1(0,1)$ while $2/3 \not\in C_1(0,1)$. And vice versa, $1/3 \not\in C_2(0,1)$, $2/3 \in C_2(0,1)$. The standard Cantor set is the sum $\tilde{C} = C_1(0,1) \cup C_2(0,1)$. All irrational elements of $\tilde{C}$ belong to $C_2(0,1)$ (an irrational number has a unique binary form), so $\tilde{C}$ and $C_2(0,1)$ differ on a countable set. Notice further that $0 \in C_2(0,1)$, with $f_2(0) = 0$. In [21,22] we worked with $C_2(0,1)$ so let us concentrate on this case. Let $C_2(k,k+1)$, $k \in \mathbb{Z}$, be the copy of $C_2(0,1)$ but shifted by $k$. We construct a fractal $X = \cup_{k \in \mathbb{Z}} C_2(k,k+1)$, and the bijection $f: X \rightarrow \mathbb{R}$. Explicitly, if $x \in C_2(0,1)$, then $x + k \in C_2(k,k+1)$, and $f(x + k) = f(x) + k$ by definition. In [21,22] the set $X$ is termed the Cantor line, and $f$ is the Cantor-line function. For more details see [21]. The set $X \setminus \{k+1\}$ is self-similar, but $X$ as a whole is not-self similar. Fig. 1 (upper) shows the plot of $g = f^t$. For completely irregular generalizations of the Cantor line, see [22].

Let us make a remark that in the literature one typically considers Cantor sets $\tilde{C}$ so that the resulting function $g: \tilde{C} \rightarrow [0,1)$ is non invertible on a countable subset. In [3] one employs the map
Fig. 1. $f^{-1}$ for the ternary Cantor line $X$ (upper), and $f^{-1}$ for the quaternary Cantor set $X = f^{-1}([0,1))$ (lower). $f$ and $f_*$ are used in construction of non-Diophantine arithmetic and calculus on both Cantor sets.

3. Non-Diophantine arithmetic of complex numbers

The examples discussed in the present paper will employ real-valued, sine and cosine Fourier transforms. However, in mind future applications it will pay to discuss in detail the construction of a complex-valued transform. In order to do so, we have to explain what should be meant by a complex number if non-Diophantine arithmetic is in use. We will follow the strategy from [21].

From now on the numbers from $X$ will be denoted by uppercase letters: $X \in \mathbb{X}, X \oplus Y \in \mathbb{X}$, and so on. The elements of $\mathbb{R}$ will be generally denoted by lower-case symbols, eg. $f(X) = x$, with very few non-ambiguous exceptions, such as $n! = f^{-1}(n!)$ instead of the apparently more consistent $n! = f^{-1}(n!)$. Non-Diophantine complex numbers, denoted by $\mathbb{C}$, will be identified with pairs of elements from $X$, subject to the following arithmetic:

\[ A \oplus B = (A_1, A_2) \oplus (B_1, B_2) \]
\[ = (A_1 \oplus B_1, A_2 \oplus B_2). \]  \tag{14}
\[ A \odot B = (A_1, A_2) \odot (B_1, B_2) \]
\[ = (A_1 \odot B_1 \odot A_2 \odot B_2, A_1 \odot B_2 \odot A_2 \odot B_1). \]  \tag{15}
and conjugation
\[ A^* = (A_1, \odot A_2). \]  \tag{16}

The modulus is defined by
\[ |A| = A \oplus A^* = (A_1^2 \oplus A_2^2, 0') \equiv A_1^2 \oplus A_2^2. \]  \tag{17}

We simplify the notation by identifying $(A_1, 0')' \in \mathbb{C}$ with $A_1 \in \mathbb{X}$. The “imaginary unit” is defined as $i = (0', 1')$, and satisfies $i \odot i = i^2 = (\odot 1', 0') \equiv \odot 1'$. We do not risk any ambiguity if we write $iA$ instead of $i \odot A$, for any $A \in \mathbb{X}$.

\[ A_1 \oplus iA_2 = (A_1, 0') \oplus i(A_2, 0') \]
\[ = (A_1, 0') \oplus (0', 1') \oplus (A_2, 0') \]
\[ = (A_1, 0') \oplus (0', A_2) = (A_1, A_2). \]  \tag{20}

Complex exponent is defined as
\[ \exp(i \phi) = (\cos \phi, \sin \phi) \]
\[ = \cos \phi \oplus i \sin \phi. \]  \tag{21}
\[ = f^{-1}(\cos f(\phi)) \oplus i f^{-1}(\sin f(\phi)) \]
\[ = f^{-1}(\cos i f(\phi)) \oplus i f^{-1}(\sin i f(\phi)) \]  \tag{22}
where
\[ \cos X = f^{-1}(\cos f(X)), \]
\[ \sin X = f^{-1}(\sin f(X)). \]  \tag{23}
The trigonometric identity reads

\[ 1' = \cos^2 X \oplus \sin^2 X \]

\( = \exp(i\phi) \circ \exp(i\phi)^* \) \hspace{1cm} (27)

\[ = \exp(i\phi) \circ \exp(ciz\phi). \] \hspace{1cm} (28)

In Taylor expansions we need a non-Diophantine factorial

\[ n'! = 1' \cdot 2' \cdot 3' \cdots n' \] \hspace{1cm} (29)

\[ = f^{-1}\left( f(1') f(2') f(3') \cdots f(n') \right) \] \hspace{1cm} (30)

\[ = f^{-1}(1 \cdot 2 \cdots n) = f^{-1}(n!). \] \hspace{1cm} (31)

Taylor expansions of elementary functions occur automatically,

\[ \cos X = f^{-1}(\cos f(X)) \] \hspace{1cm} (32)

\[ = f^{-1}\left( 1 - \frac{(f(X))^2}{2!} + \frac{(f(X))^4}{4!} - \cdots \right) \] \hspace{1cm} (33)

\[ = f^{-1}\left( f(1') - f(f(X)^2)/f(2!') + \cdots \right) \] \hspace{1cm} (34)

\[ = 1' \circ X^2 \oplus 2' \circ X^4 \oplus 4' \cdots \] \hspace{1cm} (35)

\[ = \oplus_{k=0}^{\infty}(1^{(2k)}) X^{(2k)} \circ (2k)!' \] \hspace{1cm} (36)

\[ \sin X = f^{-1}(\sin f(X)) \] \hspace{1cm} (37)

\[ = X \circ X^2 \oplus 3' \circ X^4 \oplus 5' \cdots \] \hspace{1cm} (38)

\[ = \oplus_{k=0}^{\infty}(1^{(2k)}) X^{(2k+1)} \circ (2k+1)!' \] \hspace{1cm} (39)

\[ \exp X = f^{-1}(\exp f(X)) \] \hspace{1cm} (40)

\[ = 1' \circ X \oplus X^2 \oplus 2' \circ X^3 \oplus 3' \cdots \] \hspace{1cm} (41)

\[ = \oplus_{k=0}^{\infty} X^k \circ k!'. \] \hspace{1cm} (42)

4. Non-Diophantine derivatives and integrals

A derivative of a function \( f \) is defined by

\[ \frac{DA(X)}{DX} = \lim_{h \to 0} \left( A(X \oplus H) \circ A(X) \right) \oplus H, \]

and an integral is an inverse of the derivative, so that the fundamental laws of calculus relating integration and differentiation remain valid in \( \Box \).

For example, let \( A(X) = X^N = f^{-1}(f(X)^N) \). Directly from definition (44), and taking into account \( f(N') = N \), one finds

\[ \frac{DX^N}{DX} = f^{-1}\left( N f(X)^{N-1} \right) \] \hspace{1cm} (43)

\[ = f^{-1}\left( f(N') f(X)^{N-1} \right) \] \hspace{1cm} (44)

\[ = N' \circ X^{N-1} = N' \circ X^{N-1} \oplus 1'. \] \hspace{1cm} (45)

One similarly verifies

\[ \frac{D \sin (K \circ X)}{DX} = K \circ \cos (K \circ X), \] \hspace{1cm} (46)

\[ \frac{D \cos (K \circ X)}{DX} = \cos K \circ \sin (K \circ X), \] \hspace{1cm} (47)

\[ \frac{D \exp (K \circ X)}{DX} = K \circ \exp (K \circ X), \] \hspace{1cm} (48)

\[ \frac{D \exp (iK \circ X)}{DX} = iK \circ \exp (iK \circ X). \] \hspace{1cm} (49)

A derivative of a function \( a : \mathbb{R} \to \mathbb{R} \) is defined with respect to the lowercase arithmetic,

\[ \frac{da(x)}{dx} = \lim_{h \to 0} \left( a(x+h) - a(x) \right)/h. \] \hspace{1cm} (50)

Now let \( A = f^{-1} \circ a \circ f \). Then,

\[ \frac{DA(X)}{DX} = f^{-1}(\frac{df(A)}{dX}). \] \hspace{1cm} (51)

\[ \int_X A(X')dX' = f^{-1}\left( \int_{f(X)}^{f(Y)} a(x)dx \right). \] \hspace{1cm} (52)

satisfy

\[ \frac{D}{DX} \int_Y A(X')dX' = A(X) \oplus A(Y). \] \hspace{1cm} (53)

\[ \int_Y \frac{DA(X')}{DX}dX' = A(X) \circ A(Y). \] \hspace{1cm} (54)

Formula (53) follows directly from the definitions of \( D/DX \) and \( d/dx \). In the sequel \([27]\), we show how to generalize the calculus to functions \( A : \mathbb{R} \to \mathbb{Y} \), where the sets \( \mathbb{X} \) and \( \mathbb{Y} \) are equipped with different arithmetics.

It is extremely important to realize that (53) is not the usual formula relating derivatives of \( A = f^{-1} \circ a \circ f \) and \( a \). Indeed,

\[ \frac{DA}{DX} = f^{-1} \circ \frac{da}{dx} \circ f, \] \hspace{1cm} (55)

so that \( D/DX \) behaves like a covariant derivative, but with a trivial connection. Yet, \( f \) can be any bijection \( f : \mathbb{X} \to \mathbb{R} \). The usual approach, employed in differential geometry or gauge theories, would employ the arithmetic of \( \mathbb{R} \), and one would have to assume differentiability of \( f \) and \( f^{-1} \). Here bijectivity is enough since no derivatives of either \( f \) or \( f^{-1} \) will occur in (53) and (57).

5. Scalar product

Let \( A_k, B_k : \mathbb{X} \to \mathbb{X} \), \( k = 1, 2 \). \( A_k = f^{-1} \circ a_k \circ f \), \( B_k = f^{-1} \circ b_k \circ f \), and \( A = A_1 \oplus iA_2 \). \( a = a_1 + ia_2 \). Define

\[ (A|B) = \int_{T \oplus T'} A(X) \circ B(X)DX. \] \hspace{1cm} (56)

Employing (17), (18), (54) we transform (58) into

\[ (A|B) = f^{-1}\left( \int_{f(T)/2}^{f(T)/2} \frac{df(A)}{dX}b(x)dx \right) \] \hspace{1cm} (57)

\[ \oplus i \int_{f(T)/2}^{f(T)/2} \frac{df(A)}{dX}b(x)dx. \] \hspace{1cm} (58)

\( f(T) \) can be finite or infinite. It is useful to denote \( (a|b) = \int_{f(T)/2}^{f(T)/2} \frac{df(A)}{dX}b(x)dx \), so that

\[ (A|B) = f^{-1}(\int_{f(T)/2}^{f(T)/2} \frac{df(A)}{dX}b(x)dx) \oplus i \int_{f(T)/2}^{f(T)/2} \frac{df(A)}{dX}b(x)dx. \] \hspace{1cm} (59)

In the Appendix we prove that

\[ (A|B)^\dagger = \langle B|A \rangle. \] \hspace{1cm} (60)

\[ (A|B) = A | B, \quad (A|B) = A \circ B, \quad A \in \mathcal{C}. \] \hspace{1cm} (61)
6. Fourier transform

Let \( A : \mathbb{X} \to \mathcal{D} \). The Fourier transform \( \hat{A} : \mathbb{X} \to \mathcal{D} \) is defined by

\[
\hat{A}(K) = \left( \hat{A}_1(K), \hat{A}_2(K) \right) = \hat{A}_1(K) \oplus i \hat{A}_2(K) \tag{64}
\]

\[
= \int_{\mathcal{O}^2 \varnothing^2} A(X) \circ \text{Exp}(\oplus i K \otimes X) \, DX. \tag{65}
\]

After some computations one finds its equivalent explicit form

\[
\hat{A}(K) = f^{-1} \left( \| \int_{f(T)/2}^{f(T)/2} a(x)e^{-ij(K)x} \, dx \right)
\]

\[
\oplus i \left( \int_{-f(T)/2}^{-f(T)/2} a(x)e^{-ij(K)x} \, dx \right), \tag{66}
\]

Now assume \( f(T) < \infty \). Dirac’s delta function in the space of square-integrable functions \( a : \mathbb{C} \to \mathbb{C}, \int_{-f(T)/2}^{f(T)/2} |a(x)|^2 \, dx < \infty \), can be written as

\[
\delta(x - y) = \frac{1}{f(T)} \sum_{n \in \mathbb{Z}} e^{2\pi n (x - y) / f(T)} \tag{67}
\]

\[
= \frac{1}{f(T)} + 2 \frac{1}{f(T)} \sum_{n > 0} (\cos \frac{2\pi n x}{f(T)} \cos \frac{2\pi n y}{f(T)} + \sin \frac{2\pi n x}{f(T)} \sin \frac{2\pi n y}{f(T)}), \tag{68}
\]

Denoting

\[
c_n(y) = \sqrt{\frac{2}{f(T)}} \cos \frac{2\pi n y}{f(T)}, \quad n > 0 \tag{69}
\]

\[
s_n(y) = \sqrt{\frac{2}{f(T)}} \sin \frac{2\pi n y}{f(T)}, \quad n > 0 \tag{70}
\]

\[
c_0(y) = \sqrt{\frac{1}{f(T)}}, \tag{71}
\]

\[
s_0(y) = 0 \tag{72}
\]

\[
s_n(x) = f^{-1} \left( c_n(f(X)) \right), \tag{73}
\]

\[
s_n(x) = f^{-1} \left( s_n(f(X)) \right), \tag{74}
\]

one finds

\[
\delta(x - y) = \sum_{n \in \mathbb{Z}} \left( c_n(x)c_n(y) + s_n(x)s_n(y) \right), \tag{75}
\]

which implies

\[
A(X) = \bigoplus_{n \in \mathbb{Z}} \left( c_n(X) \odot (c_n|A) \oplus s_n(X) \odot (s_n|A) \right). \tag{76}
\]

Since

\[
\langle c_n|c_m \rangle = f^{-1}(\delta_{nm}) \oplus i \left( \delta_{nm} \right), \tag{77}
\]

\[
= f^{-1}(\delta_{nm}) \oplus i f^{-1}(0) \tag{78}
\]

\[
= \delta'_{nm} \oplus i \delta' \tag{79}
\]

\[
= \langle s_n|s_m \rangle, \tag{80}
\]

\[
= \langle c_n|s_m \rangle = 0', \tag{81}
\]

\[
\langle c_n|s_m \rangle = 0'. \tag{82}
\]

where

\[
\delta'_{nm} = f^{-1}(\delta_{nm}) = \begin{cases} 1' & \text{for } n = m, \\ 0' & \text{for } n \neq m. \end{cases} \tag{83}
\]

we arrive at the Parseval formula

\[
\langle A|B \rangle = \int_{\mathcal{O}^2 \varnothing^2} A(X)^* \circ B(X) \, DX \tag{84}
\]

\[
= \bigoplus_{n \geq 0} \left( \langle A|c_n \rangle \circ \langle c_n|B \rangle \oplus \langle A|s_n \rangle \circ \langle s_n|B \rangle \right). \tag{85}
\]

7. Example: Cantoroid sawtooth function

Let us consider the sawtooth function \( a : \mathbb{R} \to \mathbb{R} \) and its ternary Cantor-line analogue, \( A = f^{-1} \circ a \circ f \), depicted in Fig. 2. Now let us perform the Fourier transform with \( f(T) = 1 \), i.e. \( T = 1 \). Fig. 3 shows two finite-sum reconstructions of \( A \).

\[
A(X) = \bigoplus_{0 \leq n \leq m} \left( c_n(X) \odot (c_n|A) \oplus s_n(X) \odot (s_n|A) \right), \tag{86}
\]

with \( m = 5 \) and \( m = 30 \) Fourier terms, respectively. The Gibbs phenomenon is clearly visible. In [27] we show how to construct a Fourier transform of a function \( A : \mathbb{X} \to \mathcal{Y} \) where \( \mathcal{Y} = \mathbb{R} \) and \( \mathbb{X} \) is a Sierpiński set.

8. Spectrum of frequencies

The Laplacian \( \Delta = \frac{\partial^2}{\partial x^2} \) satisfies

\[
\Delta c_n(X) = \frac{D}{Dx} \frac{D}{Dx} f^{-1}(c_n(f(X))) \tag{87}
\]

\[
= f^{-1} \left( -\frac{(2\pi n)^2}{f(T)^2} \sqrt{\frac{2}{f(T)}} \cos \frac{2\pi n x}{f(T)} \right) \tag{88}
\]

\[
= \circ f^{-1}(n)^2 \circ f^{-1} \left( \frac{(2\pi n)^2}{f(T)^2} \right) \odot c_n(X). \tag{89}
\]

Spectrum in the sense of Jorgensen and Pedersen [18] corresponds to \( f(T) = 1 \) and is given by \( \lambda s \) that parametrize the exponent
The concepts of differentiation and integration can be easily defined for fractals equipped with intrinsic non-Diophantine arithmetic. Once we know how to integrate and differentiate in a way that preserves the fundamental theorems of calculus, we can easily define Fourier transforms that possess all the standard properties (resolution of unity, Parseval theorem, Gibbs effect,...). Accordingly, there is no problem with momentum representation in quantum mechanics on fractal space-times, at least in the class of fractals that satisfy our assumptions. In the sequel to the present paper, we show that the class includes Sierpiński sets [27].
\[ f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right) \oplus f^{-1}\left( \Im \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right). \]

Analogously
\[ (103) = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right) \oplus f^{-1}\left( \Im \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right). \]

implying
\[ \{A|B \oplus C\} = \{A|B\} \oplus \{A|C\}. \]

Next, let \( \Lambda \in \mathbb{C} \) be a constant. Then
\[ B'(X) = (B'_1(X), B'_2(X)) \]
\[ = \Lambda \odot B(X) \]
\[ = (\Lambda_1 \oplus i\Lambda_2) \odot (B_1(X) \oplus iB_2(X)) \]
\[ = (\Lambda_1 \odot B_1(X) \oplus \Lambda_2 \odot B_2(X)) \]
\[ \oplus i'\left( \Lambda_1 \odot B_2(X) \oplus \Lambda_2 \odot B_1(X) \right) \]
\[ = f^{-1}\left( f(\Lambda_1)b_1[f(X)] - f(\Lambda_2)b_2[f(X)] \right) \]
\[ \oplus i'e^{-1}\left( f(\Lambda_1)b_2[f(X)] + f(\Lambda_2)b_1[f(X)] \right). \]

\[ B' = (B'_1, B'_2) \]
\[ = \left( f^{-1} \circ f(\Lambda_1)b_1 - f(\Lambda_2)b_2 \right) \circ f, \]
\[ f^{-1} \circ f(\Lambda_1)b_2 + f(\Lambda_2)b_1 \circ f \]
\[ = f^{-1} \circ b_1' \circ f, f^{-1} \circ b_2' \circ f \]
\[ = f^{-1} \circ \Re\left( f(\Lambda_1) + if(\Lambda_2) \right)(b_1 + ib_2) \circ f, \]
\[ f^{-1} \circ \Im\left( f(\Lambda_1) + if(\Lambda_2) \right)(b_1 + ib_2) \circ f \]
\[ \Rightarrow b_1' = \Re\left( f(\Lambda_1) + if(\Lambda_2) \right)(b_1 + ib_2), \]
\[ b_2' = \Im\left( f(\Lambda_1) + if(\Lambda_2) \right)(b_1 + ib_2). \]
\[ b' = b'_1 + ib'_2 \]
\[ = \left( f(\Lambda_1) + if(\Lambda_2) \right)(b_1 + ib_2) \]

Recall that
\[ \{A|B'\} = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b'(x)dx}{a(x)c(x)dx} \right) \]
\[ \oplus i'e^{-1}\left( \Im \int_{f(T/2)} f^{(T/2)} \frac{a(x)b'(x)dx}{a(x)c(x)dx} \right). \]

Therefore
\[ \{A|\Lambda \odot B\} = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b'(x)dx}{a(x)c(x)dx} \right) \]

\[ = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right) \]
\[ \oplus i'e^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right). \]

\[ = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right) \]
\[ \oplus i'e^{-1}\left( \Im \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right). \]

\[ = f^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right) \]
\[ \oplus i'e^{-1}\left( \Re \int_{f(T/2)} f^{(T/2)} \frac{a(x)b(x)dx}{a(x)c(x)dx} \right). \]

\[ \Rightarrow \{A|\Lambda \odot B\} \circ \Lambda = \{A|B\} \circ \Lambda \]

which ends the proof.

References