EPR-Bell averages and entanglement with vacuum in alternative approaches to field quantization

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"Entanglement with vacuum" (just to focus attention)

\[ |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \]
In brief: There are at least three different representations of a quantum vacuum state

1. A highly entangled „cyclic vector“ (algebraic QFT)

2. An unentangled but infinite tensor product of oscillator ground states (cavity QED, quantum optics)

3. A Bose-Einstein condensate of a finite oscillator gas with no relation between a number of modes and a number of oscillators (a heretical approach, MC 2000)

Tensor-product structures are here completely different!

• Does the Bell inequality feel the difference between the tensor structures?
• An „entanglement with vacuum“ means entanglement with exactly WHAT?
Construction #1: Vacuum á la algebraic QFT

Requirements

- With each subsystem we can associate local observables
- All states of the Universe can be generated by operators associated with a single subsystem (we can generate states localized in another galaxy by acting on vacuum with operators localized in Wrocław)
- That such a vacuum exists is an axiom of algebraic QFT

A consequence

- Such a vacuum violates the Bell inequality (Werner, Summers, 1985)

How to understand it?
A toy model (a two-qubit Universe)

\[ |\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |0_1\rangle \otimes |1_2\rangle \pm |1_1\rangle \otimes |0_2\rangle \right) \]

\[ |\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |0_1\rangle \otimes |0_2\rangle \pm |1_1\rangle \otimes |1_2\rangle \right) \]

Bell basis

Algebra of local operators associated with only one subsystem is enough to generate the Hilbert space

\[ A = |0_1\rangle \langle 0_1| - |1_1\rangle \langle 1_1| \]
\[ B = |0_1\rangle \langle 1_1| + |1_1\rangle \langle 0_1| \]

Vacuum in the sense of AQFT can be given by any maximally entangled state

\[ |\Omega\rangle = |\Psi_+\rangle \]

\[ (1 \otimes 1)|\Omega\rangle = |\Psi_+\rangle \]
\[ (A \otimes 1)|\Omega\rangle = |\Psi_-\rangle \]
\[ (B \otimes 1)|\Omega\rangle = |\Phi_+\rangle \]
\[ (AB \otimes 1)|\Omega\rangle = |\Phi_-\rangle \]
Moral #1:

- Vacuum state is here entangled so it's not surprising it can violate the Bell inequality
- It is not clear how to entangle anything with such a vacuum
- In algebraic QFT community this type of entangled vacuum is treated as the vacuum
Construction #2: Vacuum as an infinite tensor product of „modes”

\[ [a_m, a_n^\dagger] = \delta_{nm} \mathbf{1} \]

Algebra of a countable number of noninteracting oscillators

\[ a_n = \ldots 1 \otimes 1 \otimes a \otimes 1 \otimes 1 \otimes 1 \ldots \]

\[ 1 = \ldots 1 \otimes 1 \otimes 1 \otimes 1 \ldots \]

Vacuum represents the „ground state” of the oscillators
Such a vacuum is unentangled

\[ |0\rangle = \ldots |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \ldots \]
However, any superposition of 1-particle different modes

\[(a_n^\dagger + a_m^\dagger)|0\rangle = \ldots |0\rangle \otimes |1\rangle \otimes \ldots \otimes |0\rangle \otimes |0\rangle \ldots + \ldots |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \otimes |1\rangle \ldots\]

is similar to an entangled-with-vacuum 2-qubit state

\[|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle\]

and one can find observables that violate the Bell inequality

There exist cryptographic systems based on this property
Moral #2:

- Vacuum state is here a product (unentangled) state
- Any 1-particle superposition of different modes is „entangled with vacuum”
- It is not clear how to proceed if the number of modes is not countable (in free space, say)
- In quantum optics/quantum information community this unentangled vacuum is treated as the vacuum

Note:

- In practical quantum optical computations of EPR-Bell correlations one does not need the explicit infinite-tensor-product representation of states: Commutation relations + annihilation of vacuum by annihilation operators are enough
Construction #3: Vacuum as a Bose-Einstein condensate of a finite oscillator gas (MC 2000, 2013)

\[ |0\rangle = |O\rangle \otimes \cdots \otimes |O\rangle \]

\[ |O\rangle = \int d\tilde{k} O(\tilde{k}) |0_0, 0_1, 0_2, 0_3, k\rangle \]

Tensor product of a finite number of wave packets

Wave function of a non-monochromatic vacuum (effective IR & UV cutoff)

A single 4-dimensional oscillator wave packet with internal (oscillator) degrees of freedom in ground states

Vacuum is here represented by an infinite-dimensional Hilbert space of states annihilated by all annihilation and number operators (there is no unique BEC even at zero temperature)
Motivation

Quantize fields in a representation (of appropriate algebras) that lead to „sensible mathematics” (Dirac).

Choose a representation (or algebra) that may (perhaps) involve new degrees of freedom which have no counterpart in classical physics (Dirac).

Finally, compare with experiments.
The origin of the main idea

In physical harmonic oscillators (nano-pendulum, charge in a magnetic field) the „parameters” $\omega$ are in fact eigenvalues of some operators.

But in quantum fields they are indeed parameters.

So quantum-field oscillators are more classical than typical quantum mechanical oscillators.
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Q: What if we thus additionally quantize these classical objects implicit in quantum fields?
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Q: What if we thus additionaly quantize these classical objects implicit in quantum fields?

A: The representation of vacuum changes...
A „remarkable representation“ of Harmonic Oscillator Lie Algebra \((N=1)\)

\[
\begin{align*}
[a_a(\mathbf{k}), a_b(\mathbf{k}')\dagger] & = \delta_{ab}\delta(\mathbf{k}, \mathbf{k}')I(\mathbf{k}) \\
[a_a(\mathbf{k}), n_b(\mathbf{k}')] & = \delta_{ab}\delta(\mathbf{k}, \mathbf{k}')a_a(\mathbf{k}) \\
[a_a(\mathbf{k})\dagger, n_b(\mathbf{k}')] & = -\delta_{ab}\delta(\mathbf{k}, \mathbf{k}')a_a(\mathbf{k})\dagger
\end{align*}
\]

\[
\int d\tilde{k} I(\mathbf{k}) = I
\]

\[
\langle 0|I(\mathbf{k})|0\rangle = |O(\mathbf{k})|^2
\]

Note that the standard case would be reconstructed if

\[
\langle 0|I(\mathbf{k})|0\rangle = 1
\]

But this would be incompatible with the subsidiary condition which I impose on the central element

This is a representation for a single 4D (relativistic) harmonic oscillator. Now we need \(N\) such (bosonic) oscillators.
For $N>1$

Bosonic extension to $N$ noninteracting 4D oscillators

$$I(k, N) = \frac{1}{N} \left( I(k, 1) \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes I(k, 1) \right)$$

$$a_a(k, N) = \frac{1}{\sqrt{N}} \left( a_a(k, 1) \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes a_a(k, 1) \right)$$

$$n_a(k, N) = n_a(k, 1) \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes n_a(k, 1)$$

Vacuum is a product state

$$|0\rangle = |O\rangle \otimes \cdots \otimes |O\rangle = |0, N\rangle$$

but a single-photon state is highly entangled

$$a_a(k, N)\dagger |0, N\rangle$$

It is \textbf{not} an EPR state!
The tensor-product structure of states is completely different from the usual one...

So, how to compute EPR correlations of linear polarizations?
Arbitrary linear polarizations

\[ V_\theta(1) = I \otimes \int dk \langle k | k \rangle \otimes e^{i \theta(k)} (a_1^\dagger a_2 - a_2^\dagger a_1) \]

\[
V_\theta(1)a_1(k, 1)V_\theta(1)^\dagger = a_1(k, 1) \cos \theta(k) - a_2(k, 1) \sin \theta(k) = a_\theta(k, 1), \\
V_\theta(1)a_2(k, 1)V_\theta(1)^\dagger = a_2(k, 1) \cos \theta(k) + a_1(k, 1) \sin \theta(k) = a_{\theta'}(k, 1)
\]

For N>1

\[ V_\theta(N) = V_\theta(1) \otimes^N, \]

\[
V_\theta(N)a_1(k, N)V_\theta(N)^\dagger = a_1(k, N) \cos \theta(k) - a_2(k, N) \sin \theta(k) = a_\theta(k, N), \tag{259}
\]

\[
V_\theta(N)a_2(k, N)V_\theta(N)^\dagger = a_2(k, N) \cos \theta(k) + a_1(k, N) \sin \theta(k) = a_{\theta'}(k, N), \tag{260}
\]

Number operators defined via the same unitary transformation

\[
V_\theta(N)n_1(k, N)V_\theta(N)^\dagger = n_\theta(k, N), \\
V_\theta(N)n_2(k, N)V_\theta(N)^\dagger = n_{\theta'}(k, N)
\]

Now yes-no observables can be defined
Yes-no observables for linear polarizations

\[ Y_\theta (k, N) = n_\theta (k, N) - n_{\theta'} (k, N) \]

In practice one integrates over certain sets of wave vectors

\[ Y_\alpha (N) = \int_{\Omega} dl \ Y_\alpha (l, N), \]
\[ Y'_{\beta} (N) = \int_{\Omega'} dl' \ Y_{\beta} (l', N) \]

Such operators will be used in EPR-Bell averages
EPR field operator

$$\Psi(N) = \int dk \, dk' \, \psi(k, k') a(+, k, N)^\dagger a(-, k', N)^\dagger$$

$$\psi(k, k') = -\psi(k', k)$$

EPR state $$\Psi(N)|0, N\rangle$$

EPR-Bell average

$$\frac{\langle 0, N|\Psi(N)^\dagger Y'_{\beta}(N)Y_{\alpha}(N)\Psi(N)|0, N\rangle}{\langle 0, N|\Psi(N)^\dagger \Psi(N)|0, N\rangle}$$
Example: Non-overlapping detectors \( \Omega \cap \Omega' = \phi \)

\[
\frac{\langle 0, N | \Psi(N) \dagger Y'_\beta(N) Y_\alpha(N) \Psi(N) | 0, N \rangle}{\langle 0, N | \Psi(N) \dagger \Psi(N) | 0, N \rangle} = -\cos 2(\alpha - \beta) p_{\Omega \times \Omega'}
\]

This multiplier does not depend on \( N \), if \( N>1 \)

\[
p_{\Omega \times \Omega'} = p_{\Omega' \times \Omega}
\]

\[
= 2 \frac{\int_{\Omega} dl \int_{\Omega'} dl' |\psi(l, l')|^2 |O_0(l)|^2 |O_0(l')|^2}{\int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dk' |\psi(k, k')|^2 |O_0(k)|^2 |O_0(k')|^2}
\]

and is determined both by the form of the EPR state and the form of vacuum.

However....
Take

\[ \psi(k, k') = f(k)g(k') - f(k')g(k) \]

and

\[
\begin{align*}
\text{supp}(f) & \subset \Omega \\
\text{supp}(g) & \subset \Omega'
\end{align*}
\]

Detectors measure the whole range of the wave packets

Then

\[ p_{\Omega \times \Omega'} = 1 \]

and the EPR-Bell average is the standard one no matter what vacuum wave function \( O_0(k) \) one takes!
Moral #3

- 1-particle states have a structure of entangled states of \( N \) qubits
- Bell average for qubits defined via polarization of photons and evaluated in 2-particle „EPR states” has essentially a standard-looking form for any \( N>1 \)
- EPR states are created by acting on vacuum with appropriate field operators (the standard quantum optical construction)
- However, the tensor structure of EPR states involves entanglement of \( N \) qubits „with vacuum”
 Bibliography