

Nonlinear Schrödinger equation and two-level atoms

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Starting with the same form of atomic nonlinearity, Weinberg [Ann. Phys. (NY) 194, 336 (1989)] and Wódkiewicz and Scully [Phys. Rev. A 42, 5111 (1990)] obtained contradictory results concerning an evolution of the atomic inversion w in a two-level atom in Weinberg's nonlinear quantum mechanics: If the atom is initially in a ground state then either the evolution of w (1) can be linear if one uses a nonlinear generalization of the Jaynes-Cummings Hamiltonian, or (2) is always nonlinear if one uses the nonlinear Bloch equations derived from the nonlinear atomic Hamiltonian function. It is shown that the difference is rooted in inequivalent descriptions of the composite "atom-plus-field" system. The linear evolution of w results from a "faster-than-light communication" between the atom and the field. If one applies a description without the "faster-than-light telegraph" then the calculations based on a suitably modified Jaynes-Cummings Hamiltonian lead to the same dynamics of w as is found in semiclassical calculations based on Bloch equations. It is shown also that a nonlinear quantum mechanics based on a nonlinear Schrödinger equation does not possess a natural probability interpretation.

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I. INTRODUCTION

The purpose of the generalization of quantum mechanics proposed by Weinberg [1,3,4] "was not seriously to propose an alternative to quantum mechanics but only to have *some* theory whose predictions would be close to but not quite the same as those of quantum mechanics, to serve as a foil that might be tested experimentally" [5]. Next, in a series of very precise experiments Weinberg's theory was indeed tested [6], showing no observable deviations from linearity. All the experiments were based on the assumption that the hypothetical nonlinearity is of purely atomic origin. The assumption means formally that the atom is a subsystem of a greater "atom-plus-field" composite system where the atom itself evolves in a nonlinear way but the field and its interaction with the atom are described by ordinary linear quantum mechanics (LQM).

The main objective of this paper is to show that a knowledge of a form of a nonlinear Schrödinger equation describing an evolution of an atomic *wave function* is not sufficient for a unique description of the "atom-plus-field" system even if one assumes that the field and the interaction Hamiltonians are *linear* (i.e., like in LQM). It is also shown that nonlinear Schrödinger equations do not have a unique probability interpretation. None of those facts was taken into consideration in analysis of experiments testing linearity of QM.

That something important is overlooked can be illustrated by the following example. Weinberg showed in [1] that a system consisting initially of a single photon electromagnetic field and a two-level atom in a ground state performs ordinary *linear* Rabi oscillations even when the atomic Schrödinger equation contains a nonlinear term. On the other hand, Wódkiewicz and Scully [2], considering the same nonlinear-

ity with the same initial condition for the atomic inversion and using the Bloch equations, found *nonlinear* oscillations of the inversion. It turns out that the difference is rooted in nonequivalent descriptions of the composite "atom-plus-field" system. The linear evolution occurs if we use the description with an implicit "faster-than-light telegraph," whereas the nonlinear one is found if no such "telegraphs" are present.

The two descriptions are based on different physical assumptions which were discussed in detail in [7] and [8]. To make matters worse, it can be shown that there does not exist a simple alternative between the two ways of describing compound systems. In fact, even with the telegraphs eliminated there exists an *infinite* number of *inequivalent* possibilities of describing systems such as the atom and the field, if one assumes that the atom alone evolves according to some nonlinear Schrödinger equation.

II. EIGENVALUES AND THE LIKE

Just to get some flavor of the difficulties with the probability interpretation of NLQM consider the question of results of single measurements. Let \mathcal{H} be a finite-dimensional Hilbert space and \hat{H} a Hermitian operator acting in \mathcal{H} . If $H = \langle \psi | \hat{H} | \psi \rangle$ is the associated observable, the values of single measurements of H can be defined in at least three equivalent ways. Since in nonlinear QM the three options will lead to different results, we will use here different names for each of them.

(a) *Eigenvalue* E of $H = \langle \psi | \hat{H} | \psi \rangle$ is the number satisfying for some *eigenstate* the equation

$$\frac{\partial H}{\partial \psi_m} = \lambda \psi_m. \quad (1)$$

(b) *Diagonal values* are solutions of

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$$\det\left(\frac{\partial^2 H}{\partial \bar{\psi}_m \partial \psi_n} - \lambda \delta_{mn}\right) = 0. \quad (2)$$

(c) The third possibility comes from the fact that, since eigenstates form a complete orthogonal set of vectors in \mathcal{H} , any solution of the Schrödinger equation $id/dt|\psi\rangle = \hat{H}|\psi\rangle$ can be expressed as

$$\begin{pmatrix} \psi_1(t) \\ \vdots \\ \psi_N(t) \end{pmatrix} = \begin{pmatrix} \psi_1(0)e^{-i\omega_1 t} \\ \vdots \\ \psi_N(0)e^{-i\omega_N t} \end{pmatrix} \quad (3)$$

and the frequencies ω_n can be termed the *eigenfrequencies*. The three possibilities can also be used for definitions of the results of single measurements in nonlinear QM. The following example shows that they are no longer equivalent.

The so-called “simplest nonlinearity” considered in experiments designed as tests of Weinberg’s nonlinear QM corresponds to the following Hamiltonian function:

$$\hat{H} = \begin{pmatrix} E_1 + \epsilon(8p^3 - 20p^2 + 16p - 3) & 8\epsilon|\psi_1|^2|\psi_2|^2\bar{\psi}_1\bar{\psi}_2 \\ 8\epsilon|\psi_1|^2|\psi_2|^2\bar{\psi}_1\bar{\psi}_2 & E_2 + \epsilon(-p^3 + 4p^2 + 1) \end{pmatrix}, \quad (5)$$

where the state is assumed normalized and $p = |\psi_1|^2$. The matrix is Hermitian and its eigenvalues (equal to the diagonal values of H) are

$$E_{\pm} = \frac{1}{2}(E_1 + E_2 - 2\epsilon(8p^2 - 8p + 1) \pm [(E_1 - E_2)^2 - 8\epsilon\{(E_1 - E_2)(4p^3 - 6p^2 + 4p - 1) + 2\epsilon(20p^4 - 40p^3 + 28p^2 - 8p + 1)\}]^{1/2}). \quad (6)$$

The diagonal values, as opposed to eigenvalues, are in the nonlinear case *functions* and the number of them, again as opposed to eigenvalues, is always equal to the dimension of the suitable Hilbert space.

The solution of the respective nonlinear Schrödinger equation is

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \psi_1(0)e^{-i(E_1 + 2\epsilon\langle\sigma_3\rangle - \epsilon\langle\sigma_3\rangle^2)t} \\ \psi_2(0)e^{-i(E_2 - 2\epsilon\langle\sigma_3\rangle - \epsilon\langle\sigma_3\rangle^2)t} \end{pmatrix}, \quad (7)$$

where the averages in the exponents are integrals of motion. It follows that the eigenfrequencies are also state-dependent functions but differ from the diagonal values. Figure 1 shows eigenfrequencies, diagonal values, and nonlinear eigenvalues corresponding to (4) with $E_1 = 0$, $E_2 = 10$, and $\epsilon = 1$ (in dimensionless units). It must be stressed that averages in Weinberg’s NLQM are not in general bounded by eigenvalues and their number can be greater from the dimension of the Hilbert space even for arbitrarily small nonlinearities. In addition a standard theory of measurement based on linear observables corresponding to observers does not solve the problem either. The problem is that there is no unique de-

$$\langle\psi|\hat{H}_0|\psi\rangle + \epsilon \frac{\langle\psi|\sigma_3|\psi\rangle^2}{\langle\psi|\psi\rangle}, \quad (4)$$

where $\hat{H}_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$. The eigenvalues are $\lambda_+ = E_1 + \epsilon$, $\lambda_- = E_2 + \epsilon$, and, if $|E_2 - E_1| < 4|\epsilon|$, the third eigenvalue appears:

$$\lambda_0 = \frac{1}{2}\left(E_1 + E_2 - \frac{1}{8\epsilon}(E_1 - E_2)^2\right).$$

All the eigenstates are stationary. The eigenstate corresponding to λ_0 is not orthogonal to the remaining two eigenstates, which excludes the ordinary probability interpretation in terms of projectors.

The diagonal values are derived from the nonlinear Hamiltonian operator

scription of the “nonlinear system plus linear observer” composite system (cf. Sec. III and [7] and [8]).

III. A TWO-LEVEL ATOM IN NONLINEAR QM

A nonlinear Hamiltonian function tested in experiments was assumed in the form

$$H(\psi, \psi^*) = \langle\psi|\hat{H}_0|\psi\rangle + \frac{\langle\psi|\hat{\epsilon}|\psi\rangle^2}{\langle\psi|\psi\rangle}, \quad (8)$$

Energy

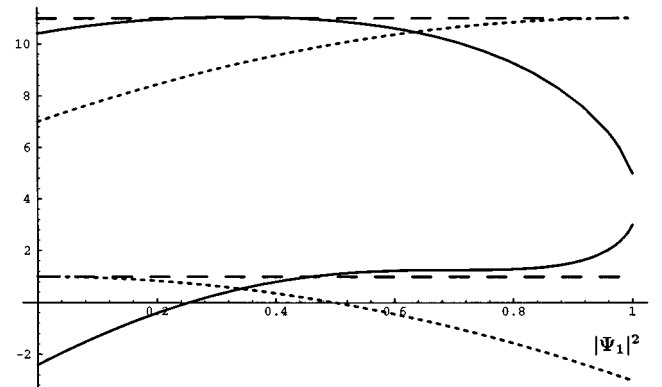


FIG. 1. Eigenvalues (dashed), eigenfrequencies (dotted), and diagonal values (solid) corresponding to (4) with $E_1 = 0$, $E_2 = 10$, and $\epsilon = 1$ (in dimensionless units), plotted as functions of $p = |\psi_1|^2$. The eigenfrequencies evaluated in eigenstates are equal to respective eigenvalues.

where $\hat{\epsilon}$ is some operator commuting with the two-level Hamiltonian \hat{H}_0 . The nonlinear evolution equation corresponding to (8) follows from the Hamilton equations given by Weinberg. These equations can be derived only in case the atom is described by a wave function which happens if the atom does not interact with the field. In case of an interaction we have to use a density matrix formalism [9]. A transition from the pure state Schrödinger equation to the density matrix formalism is not unique in NLQM.

Indeed, a density matrix form of the Hamiltonian function could involve any of the nonlinear observables

$$\frac{(\text{Tr } \rho \hat{\epsilon})^2}{\text{Tr } \rho}, \quad (9)$$

$$\frac{\text{Tr } (\rho \hat{\epsilon} \rho \hat{\epsilon})}{\text{Tr } \rho}, \quad (10)$$

$$\frac{(\text{Tr } \rho \hat{\epsilon})^2 \text{Tr } (\rho^2)}{(\text{Tr } \rho)^3}, \quad (11)$$

or

$$\frac{[\text{Tr } (\rho \hat{\epsilon} \rho \hat{\epsilon})]^{n+1}}{\text{Tr } \rho (\text{Tr } \rho \hat{\epsilon})^{2n}}, \quad (12)$$

and so on. For $\rho = |\psi\rangle\langle\psi|$, all of them, and all their convex combinations, reduce to (8), but for mixed states their properties are completely different [consider, for example, ρ satisfying $\text{Tr } \rho \hat{\epsilon} = 0$ and $\text{Tr } (\rho \hat{\epsilon} \rho \hat{\epsilon}) \neq 0$]. Each of the above choices leads to a different evolution of atomic inversion [8].

Let us choose

$$H_{\text{at}}[\rho] = \text{Tr } \rho \hat{H}_L + \frac{(\text{Tr } \rho \hat{\epsilon})^2}{\text{Tr } \rho}, \quad (13)$$

where \hat{H}_L is the linear Hamiltonian of the atom and $\hat{\epsilon}$ is an operator commuting with \hat{H}_L . Assuming that we consider the atom in a pure state $\rho = |\psi\rangle\langle\psi|$ we find that a general solution of the resulting nonlinear Schrödinger equation is (in ordinary units with $\hbar \neq 1$)

$$\psi_k(t) = \psi_k(0) \exp\left[-\frac{i}{\hbar}(E_k + 2\langle\hat{\epsilon}\rangle\epsilon_k - \langle\hat{\epsilon}\rangle^2)t\right], \quad (14)$$

where ϵ_k are eigenvalues of $\hat{\epsilon}$. The averages of $\hat{\epsilon}$ are integrals of motion and depend on *all* nonvanishing components of $|\psi\rangle$.

In the analysis of a coupling between the atom and an external electromagnetic field we meet two difficulties. First, we have to decide which states will be involved in the absorption-emission process. In linear QM the situation is simple: We take two stationary states of the noninteracting atom. In nonlinear QM the atomic nonlinearity may lead to stationary states that are not orthogonal to one another. Atomic creation and annihilation operators corresponding to such levels cannot satisfy ordinary anticommutation relations.

To avoid such complications, the interaction term we choose is defined in terms of creation and annihilation operators corresponding to the levels of the *linear* Hamiltonian.

Second, we cannot *a priori* restrict the atomic Hilbert space to two dimensions because a k th eigenfrequency depends on amplitudes of all other components of $|\psi\rangle$ and the evolution cannot be naturally “cut into N -dimensional pieces.” We cannot also assume that only these ϵ_k in which we are interested are nonvanishing. Therefore, to make the analysis perfectly consistent we should give up the two-level approximation in the interaction term. We shall not consider such complications although this approximation will further restrict the generality of the calculations presented below.

Let b_k, b_k^\dagger be the k th level atomic annihilation and creation operators satisfying the fermionic algebra $[b_k, b_l^\dagger]_+ = \delta_{kl}$ and a, a^\dagger be the annihilation and creation operators of a monochromatic photon field whose frequency is ω . The choice of the creation-annihilation operator language leads naturally to the following Hamiltonian function of the whole “atom-plus-field” composite system:

$$H_{A+F}(\psi, \bar{\psi}) = \left\langle \psi \left| \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar \omega a^\dagger a + \frac{i\hbar q}{2} (b_2^\dagger b_1 a - a^\dagger b_1^\dagger b_2) \right| \psi \right\rangle + \frac{\left\langle \psi \left| \sum_k \epsilon_k b_k^\dagger b_k \right| \psi \right\rangle^2}{\langle \psi | \psi \rangle}. \quad (15)$$

The state in the Fock basis is $|\psi\rangle = \sum_{kn} \psi_{kn} |k\rangle |n\rangle$. The nonlinear term is therefore equivalent to

$$\frac{\langle \psi | \hat{\epsilon} \otimes \mathbf{1}_F | \psi \rangle^2}{\langle \psi | \psi \rangle} = \frac{(\text{Tr } \rho_{\text{at}} \hat{\epsilon})^2}{\text{Tr } \rho_{\text{at}}}, \quad (16)$$

which is *one* out of a whole variety of inequivalent possibilities.

The Hamiltonian resulting from (15) is

$$\begin{aligned} \hat{H}_{A+F} = & \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar \omega a^\dagger a + \frac{i\hbar q}{2} (b_2^\dagger b_1 a - a^\dagger b_1^\dagger b_2) \\ & + 2 \frac{\left\langle \psi \left| \sum_k \epsilon_k b_k^\dagger b_k \right| \psi \right\rangle}{\langle \psi | \psi \rangle} \sum_k \epsilon_k b_k^\dagger b_k \\ & - \frac{\left\langle \psi \left| \sum_k \epsilon_k b_k^\dagger b_k \right| \psi \right\rangle^2}{\langle \psi | \psi \rangle^2} \end{aligned} \quad (17)$$

and the nonlinear Schrödinger equation is

$$i\hbar\dot{\psi}_{1n} = \left\{ \begin{array}{l} \hbar\omega_1 + 2 \frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \epsilon_1 - \left(\frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \right)^2 \\ + n\hbar\omega \end{array} \right\} \psi_{1n} - \frac{i}{2} \hbar q \sqrt{n} \psi_{2n-1},$$

$$i\hbar\dot{\psi}_{2n} = \left\{ \begin{array}{l} \hbar\omega_2 + 2 \frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \epsilon_2 - \left(\frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \right)^2 \\ + n\hbar\omega \end{array} \right\} \psi_{2n} + \frac{i}{2} \hbar q \sqrt{n+1} \psi_{1n+1},$$

⋮

$$i\hbar\dot{\psi}_{kn} = \left\{ \begin{array}{l} \hbar\omega_k + 2 \frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \epsilon_k - \left(\frac{\sum_{lm} \epsilon_l |\psi_{lm}|^2}{\sum_{lm} |\psi_{lm}|^2} \right)^2 \\ + n\hbar\omega \end{array} \right\} \psi_{kn} \quad \text{for } k > 2.$$

Writing $\psi_{kn} = A_{kn} \exp(-i\alpha_{kn}/\hbar)$ we find that for $k > 2$, $A_{kn} = \text{const}$ for all n . Since also $\|\psi\|$ is time independent (hereafter we put $\|\psi\| = 1$), it follows that for $k > 2$, the exponents depend on time also via $\langle \psi | \epsilon_{12} | \psi \rangle = \sum_n (\epsilon_1 |\psi_{1n}|^2 + \epsilon_2 |\psi_{2n}|^2)$ whose explicit form has to be determined. Decomposing

$$\begin{aligned} \langle \psi | \epsilon_{12} | \psi \rangle &= \frac{1}{2} \langle \psi | (\epsilon_1 + \epsilon_2) (b_1^\dagger b_1 + b_2^\dagger b_2) | \psi \rangle \\ &+ \frac{1}{2} \langle \psi | (\epsilon_1 - \epsilon_2) (b_1^\dagger b_1 - b_2^\dagger b_2) | \psi \rangle \\ &= (\epsilon_1 + \epsilon_2) \langle \psi | R_0 | \psi \rangle + (\epsilon_2 - \epsilon_1) \langle \psi | R_3 | \psi \rangle, \end{aligned} \quad (18)$$

where the first expression is an integral of motion, we see that the problem reduces to calculating $\langle \psi | R_3 | \psi \rangle$, which is one-half of the atomic inversion. Denoting $\varepsilon = \hat{\varepsilon} \otimes \mathbf{1}_F$, the total Hamiltonian can be now decomposed into two parts

$$\begin{aligned} \hat{H}_1 &= \hbar(\omega_2 - \omega_1) R_3 + 2(\epsilon_2 - \epsilon_1) \langle \psi | \varepsilon | \psi \rangle R_3 + \hbar \omega a^\dagger a \\ &+ \frac{i\hbar q}{2} (b_2^\dagger b_1 a - a^\dagger b_1^\dagger b_2), \end{aligned} \quad (19)$$

$$\begin{aligned} \hat{H}_2 &= \hbar(\omega_1 + \omega_2) R_0 + \sum_{k>2} \hbar \omega_k b_k^\dagger b_k + 2 \langle \psi | \varepsilon | \psi \rangle \left((\epsilon_1 + \epsilon_2) R_0 \right. \\ &\left. + \sum_{k>2} \hbar \epsilon_k b_k^\dagger b_k \right) - \langle \psi | \varepsilon | \psi \rangle^2. \end{aligned}$$

Operators R_3 , $R_1 = (1/2)(b_2^\dagger b_1 + b_1^\dagger b_2)$, $R_2 = (i/2)(-b_2^\dagger b_1 + b_1^\dagger b_2)$, a , and a^\dagger commute with \hat{H}_2 so that the evolution of the atomic operators R_j is generated by the following nonlinear generalization of the Jaynes-Cummings Hamiltonian:

$$\begin{aligned} \hat{H}_1 &= \hbar \omega_0 R_3 + 2 \epsilon_0 \langle \psi | \varepsilon | \psi \rangle R_3 + \hbar \omega a^\dagger a \\ &+ \frac{i\hbar q}{2} (R_+ a - a^\dagger R_-), \end{aligned} \quad (20)$$

where $\omega_0 = \omega_2 - \omega_1$ and $\epsilon_0 = \epsilon_2 - \epsilon_1$. To explicitly distinguish between initial conditions and dynamical objects we shall decompose $\langle \psi | \varepsilon | \psi \rangle$ as follows:

$$\begin{aligned} \langle \psi | \varepsilon | \psi \rangle &= \sum_{k>2,n} \epsilon_k |\psi_{kn}|^2 + (\epsilon_1 + \epsilon_2) \langle \psi | R_0 | \psi \rangle + \epsilon_0 \langle \psi | R_3 | \psi \rangle \\ &:= A + \epsilon_0 \langle \psi | R_3 | \psi \rangle, \end{aligned} \quad (21)$$

where A is an integral of motion. Denoting further $\hbar \omega_0 + 2 \epsilon_0 A = \hbar \omega'_0$, $2 \epsilon_0^2 = \hbar \varepsilon$ we finally obtain

$$\begin{aligned} \hat{H}_1 &= \hbar \omega'_0 R_3 + \hbar \varepsilon \langle \psi | R_3 | \psi \rangle R_3 + \hbar \omega a^\dagger a \\ &+ \frac{i\hbar q}{2} (R_+ a - a^\dagger R_-). \end{aligned} \quad (22)$$

We can now better understand the two-level approximation in nonlinear QM. The evolution of the atomic operators R_j is generated by a two-dimensional Hamiltonian in analogy to the linear case, but the parameters of \hat{H}_1 depend on components of the wave function corresponding to the levels being outside of the two-dimensional Hilbert space. On the other hand, the phases of the remaining components depend on the average of the atomic inversion of the two levels.

The inversion satisfies

$$\begin{aligned} \frac{d^2}{dt^2} \langle \psi | R_3 | \psi \rangle &= -q^2 \langle \psi | R_3 (\hat{N} + \frac{1}{2}) | \psi \rangle \\ &+ \frac{iq}{2} (\Delta' + \varepsilon \langle \psi | R_3 | \psi \rangle) \langle \psi | R_+ a - a^\dagger R_- | \psi \rangle, \end{aligned} \quad (23)$$

where $\hat{N} = R_3 + a^\dagger a$ and $\Delta' = \omega'_0 - \omega$. Define

$$B = \sum_{k>2,n} \hbar \omega_k |\psi_{kn}|^2 + \hbar(\omega_1 + \omega_2) \langle \psi | R_0 | \psi \rangle. \quad (24)$$

$\langle \psi | \hat{N} | \psi \rangle$, like in the linear case, is constant. In order to get rid of the average in the last line of (23) we rewrite the whole Hamiltonian function as follows:

$$\begin{aligned} H_{A+F} &= \hbar \omega_{A+F} \\ &= A^2 + B + \hbar \omega \langle \psi | \hat{N} | \psi \rangle + \hbar \Delta' \langle \psi | R_3 | \psi \rangle \\ &\quad + \epsilon_0^2 \langle \psi | R_3 | \psi \rangle^2 + \frac{i \hbar q}{2} \langle \psi | R_+ a - a^\dagger R_- | \psi \rangle. \end{aligned} \quad (25)$$

Denoting $w = 2 \langle \psi | R_3 | \psi \rangle$, $\omega_A = A^2 / \hbar$, $\omega_B = B / \hbar$, $\omega_{\text{RWA}} = \omega_{A+F} - \omega_B$ we finally get

$$\begin{aligned} \dot{w} &= 2 \Delta' (\omega_{\text{RWA}} - \omega \langle \hat{N} \rangle - \omega_A) + [\epsilon (\omega_{\text{RWA}} - \omega \langle \hat{N} \rangle - \omega_A) \\ &\quad - \Delta'] w - q^2 \langle \psi | R_3 (\hat{N} + \frac{1}{2}) | \psi \rangle - \frac{3}{4} \epsilon \Delta' w^2 - \frac{\epsilon^2}{8} w^3. \end{aligned} \quad (26)$$

This equation illustrates a characteristic inconvenience of the Poisson bracket formalism of nonlinear QM — the nonexistence of the Heisenberg picture. In the linear case we can solve the Jaynes-Cummings problem completely, independent of any particular initial conditions assumed about states. Here the term $\langle \psi | R_3 (\hat{N} + 1/2) | \psi \rangle$ involves correlations between the atom and the field and I have not managed to express it solely in terms of constants and w unless the state is an eigenstate of \hat{N} , or a semiclassical decorrelation is assumed. So let initially the state of the system be a common eigenstate of R_3 and $a^\dagger a$ with respective eigenvalues n' and n . The atomic inversion then satisfies the general elliptic equation [13]

$$\begin{aligned} \ddot{w} &= 2 \Delta' (\Delta' n' + \frac{1}{8} \epsilon) + \left(\epsilon (\Delta' n' + \frac{1}{8} \epsilon) - \Delta' - \frac{q^2}{2} (N + \frac{1}{2}) \right) w \\ &\quad - \frac{3}{4} \epsilon \Delta' w^2 - \frac{\epsilon^2}{8} w^3. \end{aligned} \quad (27)$$

Let us take the “two-level initial conditions,” that is, assume that the initial state of the system is such that the only nonvanishing components of the wave function are those with $k = 1, 2$. Then $\hbar \Delta' = \hbar \Delta + \epsilon_2^2 - \epsilon_1^2$ where $\Delta = \omega_0 - \omega$. In most of the papers dealing with two-level systems (cf. [1,2]) the authors assume the nonlinearities with $\epsilon_2 = 0$, which results in a shift of the resonant frequency. Note, however, that the (equally simple) choice of $\epsilon_1 = -\epsilon_2$ causes no shift. Choosing the “detuning” $\Delta' = 0$ and denoting $\epsilon^2/8 = 2\varsigma^2$ we get

$$\ddot{w} = (2\varsigma^2 - \Omega^2) w - 2\varsigma^2 w^3, \quad (28)$$

and, with the initial condition $w(0) = -1$, we find

$$w(t) = \begin{cases} -\text{cn}(\Omega t, \varsigma/\Omega) & \text{for } \Omega > \varsigma \\ -\text{sech}(\Omega t) & \text{for } \Omega = \varsigma \\ -\text{dn}(\varsigma t, \Omega/\varsigma) & \text{for } \Omega < \varsigma, \end{cases} \quad (29)$$

in agreement with Wódkiewicz and Scully [2] who chose the Bloch equation approach.

It is appropriate to compare our results with those of Weinberg whose description of the “atom-plus-field” is basis dependent. With the same form of the atomic nonlinearity

the Hamiltonian function of the composite system in the *Fock basis* (this basis was chosen by Weinberg) is

$$\begin{aligned} H_{A+F}(\psi, \bar{\psi}) &= \left\langle \psi \left| \sum_k \hbar \omega_k b_k^\dagger b_k + \hbar \omega a^\dagger a + \frac{i \hbar q}{2} (b_2^\dagger b_1 a \right. \right. \\ &\quad \left. \left. - a^\dagger b_1^\dagger b_2) \right| \psi \right\rangle + \sum_n \frac{\left\langle \psi \left| \sum_k \epsilon_k b_k^\dagger b_k \hat{P}_n \right| \psi \right\rangle^2}{\langle \psi | \hat{P}_n | \psi \rangle}, \end{aligned} \quad (30)$$

where \hat{P}_n project on n -photon states. If the initial state of the whole system has only one nonvanishing component ψ_{11} (one photon and the atom in the ground state) then only ψ_{11} and ψ_{20} will appear in the nonlinear Schrödinger equation. Our “nonmalignant” choice leads to a Schrödinger equation of the form

$$\begin{aligned} i \hbar \dot{\psi}_{11} &= \left\{ \hbar \omega_1 + 2 \frac{\epsilon_1 |\psi_{11}|^2 + \epsilon_2 |\psi_{20}|^2}{|\psi_{11}|^2 + |\psi_{20}|^2} \epsilon_1 \right. \\ &\quad \left. - \left(\frac{\epsilon_1 |\psi_{11}|^2 + \epsilon_2 |\psi_{20}|^2}{|\psi_{11}|^2 + |\psi_{20}|^2} \right)^2 + \hbar \omega \right\} \psi_{11} - \frac{i}{2} \hbar q \psi_{20}, \end{aligned} \quad (31)$$

$$\begin{aligned} i \hbar \dot{\psi}_{20} &= \left\{ \hbar \omega_2 + 2 \frac{\epsilon_1 |\psi_{11}|^2 + \epsilon_2 |\psi_{20}|^2}{|\psi_{11}|^2 + |\psi_{20}|^2} \epsilon_2 \right. \\ &\quad \left. - \left(\frac{\epsilon_1 |\psi_{11}|^2 + \epsilon_2 |\psi_{20}|^2}{|\psi_{11}|^2 + |\psi_{20}|^2} \right)^2 \right\} \psi_{20} + \frac{i}{2} \hbar q \psi_{11}, \end{aligned}$$

while Weinberg’s one leads to

$$\begin{aligned} i \hbar \dot{\psi}_{11} &= \left\{ \hbar \omega_1 + 2 \frac{\epsilon_1 |\psi_{11}|^2}{|\psi_{11}|^2} \epsilon_1 - \left(\frac{\epsilon_1 |\psi_{11}|^2}{|\psi_{11}|^2} \right)^2 + \hbar \omega \right\} \psi_{11} \\ &\quad - \frac{i}{2} \hbar q \psi_{20}, \end{aligned} \quad (32)$$

$$i \hbar \dot{\psi}_{20} = \left\{ \hbar \omega_2 + 2 \frac{\epsilon_2 |\psi_{20}|^2}{|\psi_{20}|^2} \epsilon_2 - \left(\frac{\epsilon_2 |\psi_{20}|^2}{|\psi_{20}|^2} \right)^2 \right\} \psi_{20} + \frac{i}{2} \hbar q \psi_{11}.$$

Simplifying the fractions we obtain

$$i \hbar \dot{\psi}_{11} = (\hbar \omega_1 + \epsilon_1^2) \psi_{11} - \frac{i}{2} \hbar q \psi_{20}, \quad (33)$$

$$i \hbar \dot{\psi}_{20} = (\hbar \omega_2 + \epsilon_2^2) \psi_{20} + \frac{i}{2} \hbar q \psi_{11},$$

which are *linear*, and the only modification with respect to ordinary QM is that the energy levels of the atom are given by *nonlinear eigenvalues* corresponding to the atom in the absence of radiation. In addition, the “ σ_3 ” nonlinearity satisfies $\epsilon_1^2 = \epsilon_2^2$ so that this nonlinearity changes neither the energy difference nor the shape of the inversion oscillation,

whereas the “correct” description (i.e., without superluminal influences [8]) leads to elliptic oscillations even for $w(0) = -1$ and $N + \frac{1}{2} = 1$.

This result explains the difference between the calculations of Weinberg and those of Wódkiewicz and Scully. Both of them are based, more or less implicitly, on different assumptions about the description of composite systems. The Weinberg description involves the basis-dependent form of the Hamiltonian function which leads to the faster-than-light communication between the atom and the electromagnetic field. The description of Wódkiewicz and Scully does not explicitly treat the atom as a subsystem of the “atom-plus-field” composite system, but the resulting Bloch equations turn out to be consistent with the density matrix formalism which is known to eliminate the “faster-than-light-telegraph.” Still, this coincidence is purely accidental since there exists, as pointed out above, an infinite number of inequivalent descriptions where the “telegraph” is absent. If we had chosen some other “correct” form of the total Hamiltonian function in our Jaynes-Cummings approach, we

would have obtained a different solution for the atomic inversion [8].

The calculations in this section have not used any assumption about the “smallness” of the nonlinearity. Moreover, as we have shown before, it is not evident what should be actually meant by a small nonlinearity. The ambiguity is related to the existence of singular nonlinearities which are negligible in the absence of correlations, but which can become dominant if the nonlinear system in question correlates with something else. However, since it is reasonable to expect that any physical nonlinearity should in ordinary situations be small in some sense, it becomes important to understand in what respect the solutions we have found depend on approximations. In particular, the role of the rotating wave approximation should be clarified. The easiest way of doing that is to consider transitions with the selection rules $\Delta m = \pm 1$ involving circularly polarized light. It can be easily shown that the only difference with respect to the $\Delta m = 0$ transitions discussed above is the necessity of substituting $q\bar{q}$ for q^2 in the equations for w , so that no qualitative change in the time dependence of w will appear.

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- [1] S. Weinberg, *Ann. Phys. (NY)* **194**, 336 (1989).
 [2] K. Wódkiewicz and M. O. Scully, *Phys. Rev. A* **42**, 5111 (1990).
 [3] S. Weinberg, *Nucl. Phys. B (Proc. Suppl.)* **6**, 67 (1989).
 [4] S. Weinberg, *Phys. Rev. Lett.* **62**, 485 (1989).
 [5] S. Weinberg, *Dreams of a Final Theory* (Hutchinsons, London, 1993).
 [6] J. J. Bollinger, D. H. Heinzen, W. M. Itano, S. L. Gilbert, and D. J. Wineland, *Phys. Rev. Lett.* **63**, 1031 (1989); T. E. Chupp and R. J. Hoare, *ibid.* **64**, 2261 (1990); R. L. Walsworth, I. F. Silvera, E. M. Mattison, and R. F. C. Vessot, *ibid.* **64**, 2599 (1990); P. K. Majumder, B. J. Venema, S. K. Lamoreaux, B. R. Heckel, and E. N. Forston, *ibid.* **65**, 2931 (1990).
 [7] M. Czachor, Ph.D. thesis, Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, 1994.
 [8] M. Czachor, Report No. quant-ph/9501007 in Los Alamos National Laboratory electronic archive; it can be reached by electronic mail at quant-ph@xxx.lanl.org
 [9] Actually, if the whole system is described by a wave function, then a subsystem (in LQM) is described by a reduced density matrix, which is some function of the original wave function (in finite dimensional Hilbert space $\rho_{mn} = \sum_k \Psi_{mk} \Psi_{nk}^*$). It was shown by Polchinski [10] that if all observables corresponding to subsystems are of the form $F(\Psi, \Psi^*) = f(\rho)$, where ρ is the reduced density matrix representing the subsystem, then such observables commute with one another if the subsystems are separated and noninteracting. This property can be generalized to situations where the whole system is described by an arbitrary density matrix [7]. It was shown in [7] that Polchinski’s form eliminates two different forms of “faster-than-light telegraphs” even though the condition of commutability of observables is not, in general, a sufficient condition for their nonexistence. Another necessary condition is that observables corresponding to separated systems have to be independent of choices of bases in the Hilbert space describing the composite system. Weinberg’s description proposed in [1] does not have this property and hence leads to the telegraph described by Gisin [11,12].
 [10] J. Polchinski, *Phys. Rev. Lett.* **66**, 397 (1991).
 [11] N. Gisin, *Phys. Lett. A* **143**, 1 (1990).
 [12] M. Czachor, in *Bell’s Theorem and the Foundations of Modern Physics—Cesena’91*, edited by A. van der Merwe and F. Selleri (World Scientific, Singapore, 1992).
 [13] H. T. Davies, *Introduction to Nonlinear Differential and Integral Equations* (Dover Publications, New York, 1977).