Reducible representations of canonical commutation relations applied to two-level atoms interacting with electromagnetic field

Reducowalne reprezentacje kanonicznych związków komutacyjnych w zastosowaniu do atomów dwupoziomowych oddziałujących z polem elektromagnetycznym

Marcin Wilczewski

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To my wonderful parents, J. and K.
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It seems useful, then, to examine the various disturbing features of QED, which give rise to mathematical or conceptual difficulties, to ask whether present empirical evidence demands their presence, and to explore the consequences of the modified (although perhaps rather crude and incomplete) theories in which these features are removed. Any difference between the predictions of QED and some alternative theory, corresponds to an experiment which might distinguish between them; if it appears untried but feasible, then we have the opportunity to subject QED to a new test in which we know just what to look for, and which we would be very unlikely to think of without the alternative theory. For this purpose, the alternative theory need not be worked out as completely as QED; it is sufficient if we know in what way their predictions will differ in the area of interest. Nor does the alternative theory need to be free of defects in all other respects; for if experiment should show that it contains just a single ‘element of truth’ that is not in QED, then the alternative theory will have served its purpose; we would have the long-missing clue showing in what way QED must be modified, and electrodynamics (and, I suspect, much more of theoretical physics along with it) could get moving again.

E. T. Jaynes at the Rochester conference Coherence and Quantum Optics III, 1972
Preface

Introduction

Interaction between quantized fields and atoms has been studied for many years now, both from theoretical and experimental points of view. Most, if not all, theoretical treatments start with the assumption that representations of algebras of harmonic oscillator and canonical commutation relations (CCR), inherent in field operators, are irreducible. The thesis concentrates mainly on exploring various consequences of defining electromagnetic field by means of a class of reducible representations of CCR.

This, at first sight modest, modification of the paradigm fundamentally alters the very structure of the field and the nature of the atom-field interaction [1, 2, 3]. Moreover, it provides a totally different interpretation of fundamental processes lying behind observable facts, as well as predicts new physical phenomena [4, 5].

The original idea behind the formalism was to quantize not only the canonical variables $q$ and $p$ of field’s oscillators, but also their frequencies $\omega$ [1], which remain classical parameters in all the conventional treatments. This is, in a sense, field quantization which goes a step deeper than the one proposed in nineteen-twenties by M. Born, W. Heisenberg, P. Jordan [6], and P. A. M. Dirac [7]. An immediate consequence is that the notion of ordinary quantum harmonic oscillator, essential for the standard formulation of quantum field theory, turns into a kind of indefinite-frequency oscillator, whose finite ensembles may be regarded as models of the quantum field. More specifically, quantum fields may be regarded as systems composed of a large number $N$ of particles, where a one-to-one correspondence between the number of field-oscillators and the number of modes is lost. This means that the number of modes is independent of the number of oscillators, and each field-oscillator becomes a wave packet in respective Hilbert space. In practice, this means that even the well-known Jaynes-Cummings model [8] needs, when seen from the reducible perspective, to be described as a two-level atom interacting with a multi-particle system (a sort of Bose-Einstein condensate at zero temperature).

Perhaps, one of the most unusual features of the reducible approach is that it seems to be far more general than that based on irreducible representations. An issue of great importance is the limiting case $N \to \infty$, in which the law of large numbers reduces the number of different field configurations to one. Under this condition, predictions of the new theory reduce to those of the standard regularized one, and agree with experiment. The peculiar point, however, is that the limit is not at all necessary to achieve agreement with experiment. In that case, $N < \infty$, the corresponding equations and their solutions remain fundamentally different from their standard counterparts. These formal differences could be experimentally observed for sufficiently long atom-field interaction times — it is a place where cavity QED enters as a natural test-ground for the proposed reformulation.

On one hand, comparing theoretical predictions with experimental data one can gradually estimate the lower bound for the parameter $N$, and on the other hand, fulfilling
certain conditions of experimental settings one could, in principle, observe phenomena that contradict conventional understanding of quantized electromagnetic field. At the moment, however, it is not possible to show any experimental proof that the representation in question is physical, as well as no one can prove the contrary.

It is common that irreducibility of a representation is a decisive criterion for selecting a representation to work with. Irreducible representations are regarded as most fundamental, and the ones that give rise to simplest possible theories. In light of the results presented in the thesis it is possible that the real situation may be much more complicated. Reducible representations seem to be a powerful and yet underestimated tool of mathematical description of the physical world.

Although the foundations of quantum field theory were laid eighty years ago, severe problems of infinities still remain unsolved. What is interesting, in all the analyzed physical scenarios the reducible representation formalism seems to automatically generate cutoff functions that remove divergences of QED, and produce renormalized coupling constants (even in places where it is not necessary). This embodies David Finkelstein’s idea of quantization as regularization [9] in the sense that additional quantization of field frequencies eliminates QED’s infinities and ad hoc regularization methods.

Structure of the work

The thesis is organized as follows. In chapter 1 we remind the very basics of historically first approach to field quantization, based on irreducible representations of CCR.

The next chapter is an introduction into conceptual as well as purely technical consequences of the use of reducible representations. We introduce the idea of an indefinite-frequency field oscillator, playing the central role in the theory of fields quantized in the reducible way. The results show that an ensemble of such oscillators can be, in principle, considered as a model of a quantum field; the number $N$ of oscillators serves as a parameter of the theory. We construct reducible analogs of vacuum, $n$-photon and coherent states, and formulate the condition which is sufficient for reconstructing the standard theory from the reducible one.

In chapter 3 we analyze the Jaynes-Cummings model [8, 10] in an ideal cavity, for both vacuum and coherent states, within irreducible and reducible frameworks. We show that all irreducible representations imply equivalent physics, while the reducible ones permit some unusual phenomena to exist. A good example are vacuum Rabi oscillations with collapses and revivals. We discuss the limit of infinite number of oscillators, $N \to \infty$, which recovers the standard formulas. This fact is of great importance, since it plays a role of a correspondence principle and guarantees that the new theory may be regarded as a generalization of the one based on irreducible representations. Chapter ends with a statement of a heuristic procedure, with the help of which one can combine standard solutions into these corresponding to the reducible theory.

Chapter 4 deals with a variety of quantum master equations. The discussion starts with explaining a misunderstanding in the famous paper by Brune et al. [11]. Then we proceed to analyze the properties and predictions of a master equation being a kind of standard for quantum opticians. The results show that it is probably too classical for setups involving
open cavities. More appropriate microscopic formulation has been recently presented in [12, 13]. The main idea is to describe dissipation in the language of dressed states rather than the bare ones. This concept finally evolved into a generalized model [14, 15] allowing for additional long-wave transitions overlooked by other models but, perhaps, playing a key role in fitting the experimental data with theoretical curves. The two types of transitions (within a single manifold, or between different manifolds of dressed states) turn out to have different physical consequences. For example, these are the ignored fluctuations within a single manifold that control separability properties of the atom–photon system. The knowledge of all these issues allows us to ask the question whether the unusual features of the reducibly quantized field could be observed in real-life cavities affected by decoherence.

Chapter 5 contains derivation of the generalized master equation for reducible representations and analysis of its predictions [15]. We show that experiments can, under certain conditions, discriminate between irreducible and reducible representations. The last chapter is left for stating some open problems.
Chapter 1

Field as an ensemble of oscillators

Source-free solutions of classical Maxwell equations for fields contained in a cubic cavity of volume $V$ can be expressed in terms of plane waves as follows

\[ E(r, t) = i \sum_{k, \lambda} \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} (\epsilon_{k, \lambda} a_{k, \lambda}(t) e^{ik \cdot r} - \epsilon_{k, \lambda}^* a_{k, \lambda}^*(t) e^{-ik \cdot r}) , \]  

(1.1)

\[ B(r, t) = i \sum_{k, \lambda} \sqrt{\frac{\hbar}{2 \epsilon_0 V}} ((k \times \epsilon_{k, \lambda}) a_{k, \lambda}(t) e^{ik \cdot r} - (k \times \epsilon_{k, \lambda}^*) a_{k, \lambda}^*(t) e^{-ik \cdot r}) . \]  

(1.2)

Planck’s constant is here introduced just for dimensional reasons and could be replaced by any constant whose dimension is energy $\times$ time. Maxwell’s equations imply that Fourier amplitudes satisfy the harmonic oscillator equations

\[ i \frac{d}{dt} a_{k, \lambda}(t) = \omega_k a_{k, \lambda}(t) , \]  

(1.3)

\[ i \frac{d}{dt} a_{k, \lambda}^*(t) = -\omega_k a_{k, \lambda}^*(t) . \]  

(1.4)

The energy of this classical field,

\[ \mathcal{E} = \frac{\epsilon_0}{2} \int_V d^3r \left( E^2 + c^2 B^2 \right) , \]  

(1.5)

after a series of mathematical operations can be reexpressed in an equivalent form

\[ \mathcal{E} = \sum_{k, \lambda} \hbar \omega_k a_{k, \lambda}^*(t) a_{k, \lambda}(t) = \sum_{k, \lambda} \left( \frac{p_{k, \lambda}(t)^2}{2m} + \frac{m \omega_k^2 q_{k, \lambda}(t)^2}{2} \right) \]  

(1.6)

\[ = \sum_{k, \lambda} \hbar \omega_k a_{k, \lambda}^*(0) a_{k, \lambda}(0) = \sum_{k, \lambda} \left( \frac{p_{k, \lambda}(0)^2}{2m} + \frac{m \omega_k^2 q_{k, \lambda}(0)^2}{2} \right) , \]  

(1.7)

where

\[ a_{k, \lambda}(t) = \frac{1}{\sqrt{2 \hbar \omega_km}} \left( m \omega_k q_{k, \lambda}(t) + ip_{k, \lambda}(t) \right) . \]  

(1.8)
The classical field energy (1.6) is reminiscent of the energy of an ensemble of independent nonrelativistic harmonic oscillators of mass \( m \), oscillating with frequencies \( \omega_k \). Of course, the parameters such as \( m \) and \( \hbar \) have no real physical meaning here as the field is classical and massless.

1.1 Born, Heisenberg, and Jordan quantization

Although the above exercise is purely formal, it shows that replacing classical canonical variables by their quantized versions one can arrive at a quantum version of electromagnetic fields. This could be done, and in fact was done by Born, Heisenberg, and Jordan [6], already in 1925, after Heisenberg had published his paper on matrix quantization of harmonic oscillators [16]. One should bear in mind that in 1925 the idea of Schrödinger that values of physical quantities are eigenvalues of operators, was not yet known. Schrödinger’s paper, where he introduced his famous equation and interpreted energy levels as eigenvalues of a Hamiltonian operator, appeared in 1926.

What Heisenberg, Born and Jordan did was to assume that \( \hat{p}_{k,\lambda}(t) \) and \( \hat{q}_{k,\lambda}(t) \) are replaced by operators \( \hat{\hat{a}}_{k,\lambda}(t) \) and \( \hat{\hat{a}}^\dagger_{k,\lambda}(t) \) satisfying the Heisenberg formula characteristic of independent oscillators

\[
[\hat{\hat{a}}_{k,\lambda}(t), \hat{\hat{a}}^\dagger_{k',\lambda'}(t)] = i\hbar \delta_{kk'} \delta_{\lambda\lambda'},
\]

but the remaining quantities they treated as parameters. So they took

\[
\hat{\hat{a}}_{k,\lambda}(t) = \frac{1}{\sqrt{2\hbar\omega_k m}} \left( m \omega_k \hat{q}_{k,\lambda}(t) + i \hat{p}_{k,\lambda}(t) \right),
\]

satisfying

\[
[\hat{\hat{a}}_{k,\lambda}(t), \hat{\hat{a}}^\dagger_{k',\lambda'}(t)] = \delta_{kk'} \delta_{\lambda\lambda'},
\]

and not, say,

\[
\hat{a}_{k,\lambda}(t) = \frac{1}{\sqrt{2\hbar\omega_k m}} \left( m \omega_k \hat{q}_{k,\lambda}(t) + i \hat{p}_{k,\lambda}(t) \right),
\]

that would lead to a commutator different from (1.11), as we shall see later.

The electromagnetic fields

\[
\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{k,\lambda} \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \left( \epsilon_{k,\lambda} \hat{a}_{k,\lambda}(t) e^{i \mathbf{k} \cdot \mathbf{r}} - \epsilon^*_{k,\lambda} \hat{a}^\dagger_{k,\lambda}(t) e^{-i \mathbf{k} \cdot \mathbf{r}} \right),
\]

\[
\hat{\mathbf{B}}(\mathbf{r}, t) = i \sum_{k,\lambda} \sqrt{\frac{\hbar}{2 \epsilon_0 V}} \left( (\mathbf{k} \times \epsilon_{k,\lambda}) \hat{a}_{k,\lambda}(t) e^{i \mathbf{k} \cdot \mathbf{r}} - (\mathbf{k} \times \epsilon^*_{k,\lambda}) \hat{a}^\dagger_{k,\lambda}(t) e^{-i \mathbf{k} \cdot \mathbf{r}} \right),
\]
when inserted into
\[
\hat{H} = \frac{\epsilon_0}{2} \int_V d^3r \left( \mathbf{E}^2 + \epsilon^2 \mathbf{B}^2 \right),
\]  
(1.15)

led them to
\[
\hat{H} = \frac{1}{2} \sum_{k,\lambda} \hbar \omega_k \left( \hat{a}_{k,\lambda}^\dagger \hat{a}_{k,\lambda} + \hat{a}_{k,\lambda} \hat{a}_{k,\lambda}^\dagger \right) = \sum_{k,\lambda} \hbar \omega_k \left( \hat{a}_{k,\lambda}^\dagger \hat{a}_{k,\lambda} + \frac{1}{2} \right),
\]  
(1.16)

Born, Heisenberg and Jordan were aware that the result they obtained is mathematically wrong, since (1.16) contains the divergent term \( \sum_{k,\lambda} \hbar \omega_k / 2 \). What they did was what later became a standard trick in practically all quantum field theory textbooks, namely they shifted all single Hamiltonian contributions downwards by their ground state energies, and thus arrived at
\[
\hat{H} = \sum_{k,\lambda} \hbar \omega_k \hat{a}_{k,\lambda} \hat{a}_{k,\lambda} = \sum_{k,\lambda} \hbar \omega_k \hat{N}_{k,\lambda},
\]  
(1.17)

which looked analogously to the classical expression. The operator \( \hat{N}_{k,\lambda} \), whose eigenvalues are \( 0, 1, 2, \ldots \), is known as the number-of-photons operator. For a physicist a problem is that the ground-state contribution to the harmonic oscillator energy is really observed in experiments. Therefore, if one believes that the field is a gas of infinitely many oscillators, its ground state energy becomes really infinite. Subtraction of infinities is in arithmetics forbidden for the same reason as division by zero.\(^1\) There exist various procedures of removing infinities from quantum field theory, but all of them essentially reduce to regularizations by cutoff functions, inserted into divergent expressions at various stages of calculations.

The eigenstates of \( \hat{N}_{k,\lambda} \) — the Fock states — are generated by repeatedly acting on the ground state with creation operators associated with particular modes,
\[
| n_{k_1,\lambda_1}, \ldots, n_{k_j,\lambda_j} \rangle = \frac{a_{n_{k_1,\lambda_1}}^\dagger \cdots a_{n_{k_j,\lambda_j}}^\dagger}{\sqrt{n_{k_1,\lambda_1}! \cdots n_{k_j,\lambda_j}!}} | 0 \rangle.
\]  
(1.18)

The corresponding eigenvalue is \( n_{k_1,\lambda_1} + \cdots + n_{k_j,\lambda_j} \).

### 1.2 The basic commutator revisited

Studying quantum optics one can get the impression that the most fundamental algebraic structure that determines all quantum properties of light is the commutation relation (1.11). From a formal point of view, however, this relation is less natural than it may seem. The constants at the right side of this equation actually mean “constants times identity”, but this notion of “identity” is not the identity with respect to the Lie product, but with respect to the product which is undefined on the level of the algebra. More precisely, if we denote
\[^1\]1 · 0 = 2 · 0; dividing both sides by 0 we get 1 = 2, which makes arithmetics ambiguous. Analogously, 1 + \( \infty \) = 2 + \( \infty \) subtracting \( \infty \) from both sides we get 1 = 2.
by \( \star \) the Lie product of two elements of the Lie algebra, i.e. \( A \star B = [A, B] \), then the identity 1 implicitly present at the right-hand side of (1.9) means \( 1C = C1 = C \) in the sense of a product which is absent in the definition of the Lie algebra, instead of \( 1 \star C = C \), for all \( C \). This simple observation leads to the conclusion that the Lie algebra of canonical commutation relations (CCR) should rather read

\[
[q_{k,\lambda}(t), \hat{p}_{k',\lambda'}(t)] = i\hbar \delta_{kk'} \delta_{\lambda\lambda'} \hat{I}_{k',\lambda'},
\]

(1.19)

or alternatively

\[
[a_{k,\lambda}(t), \hat{a}^\dagger_{k',\lambda'}(t)] = \delta_{kk'} \delta_{\lambda\lambda'} \hat{I}_{k,\lambda'},
\]

(1.20)

where \( \hat{I}_{k,\lambda} \) is in the center of the Lie algebra, that is its Lie product \( \star \) with all elements of the algebra yields zero.

From this perspective, the standard commutator (1.9) with 1 at the right side should be regarded as a particular representation of (1.19).

Irreducible representations of physical symmetries are usually associated with elementary, hence single-particle systems. Multi-particle systems are described by reducible representations since tensor products of irreducible representations are typically reducible. Moreover, we know that, according to Heisenberg, Bohr and Jordan, quantum fields are ensembles of (infinitely) many particles. It seems that it is not \( a \) priori obvious that the representations of (1.20) should be irreducible.

In order to better understand the kind of reducibility we will discuss in this work, let us consider a system described by the Hamiltonian

\[
H = \hbar a^\dagger a \otimes \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix},
\]

(1.21)

where \([a, a^\dagger] = 1\). We will later see that interaction terms of the form \( a^\dagger a \otimes B \) occur in models of field-reservoir interactions in open cavities. The Hilbert space is spanned here by the vectors

\[
|n, \omega_1\rangle = |n\rangle \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = |n\rangle \otimes |\omega_1\rangle,
\]

(1.22)

\[
|n, \omega_2\rangle = |n\rangle \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = |n\rangle \otimes |\omega_2\rangle,
\]

(1.23)

where

\[
a^\dagger a |n\rangle = n|n\rangle.
\]

An average energy evaluated in the state

\[
|\psi\rangle = \sum_{n=0}^{\infty} (\psi_{n,\omega_1} |n, \omega_1\rangle + \psi_{n,\omega_2} |n, \omega_2\rangle)
\]

(1.25)
\[ \langle \psi | H | \psi \rangle = \sum_{n=0}^{\infty} n \hbar \omega_1 |\psi_{n, \omega_1}|^2 + \sum_{n=0}^{\infty} n \hbar \omega_2 |\psi_{n, \omega_2}|^2 \]
\[ = \langle \psi | H_{\omega_1} | \psi \rangle + \langle \psi | H_{\omega_2} | \psi \rangle, \quad (1.26) \]
i.e. resembles an average energy of a system of two independent oscillators. The Hamiltonians of these oscillators can be written as
\[ H_{\omega_1} = \hbar \omega_1 a^\dagger a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ = \hbar \omega_1 \left( a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)^\dagger \left( a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \]
\[ = \hbar \omega_1 a_1^\dagger a_1, \quad (1.27) \]
\[ H_{\omega_2} = \hbar \omega_2 a^\dagger a \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ = \hbar \omega_2 \left( a \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)^\dagger \left( a \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \]
\[ = \hbar \omega_2 a_2^\dagger a_2. \quad (1.32) \]

Summing up, the Hamiltonian that is consistent with the average (1.25) can be written as
\[ H = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2, \quad (1.33) \]
and suggests that the system in question consists of two oscillators characterized by \( \omega_1 \) and \( \omega_2 \), respectively. The Lie algebra is here the following
\[ [a_1, a_1^\dagger] = 1 \otimes |\omega_1\rangle \langle \omega_1| = I_{\omega_1}, \quad (1.34) \]
\[ [a_2, a_2^\dagger] = 1 \otimes |\omega_2\rangle \langle \omega_2| = I_{\omega_2}, \quad (1.35) \]
with the remaining commutators vanishing. The elements at the right-hand sides are in the center of the algebra, but are different from the identity. Accordingly, by Schur’s lemma, the representation is not irreducible.

Systems whose averages look as if they described ensembles of particles characterized by some parameters, but whose Hilbert space is a direct sum and not a tensor product of Hilbert spaces indexed by these parameters, are well known in quantum mechanics: These are single-particle wave packets and not ensembles of particles. The difference between these two structures is the same as the one between a single Schrödinger cat, which is in a
superposition of being dead and alive, and a classical flock of cats — some of them alive, and some of them dead. In both cases the probabilities of finding a living cat can be identical.

The most interesting case is the one of a flock of Schrödinger cats. In 1925, Heisenberg, Born and Jordan treated quantized electromagnetic fields as an ensemble of different oscillators — one oscillator per mode. The ensemble had to be infinite since the number of modes was infinite, and they did not yet understand the concept of quantum superposition. The idea developed in this thesis is to investigate experimental consequences of the assumption that each of the oscillators is a wave packet, a superposition of infinitely many different states, one state per mode, but the field itself is a finite ensemble of such wave packets.
Chapter 2
Ensembles of indefinite-frequency oscillators

A simple and interesting example of a dynamical system in quantum mechanics is the harmonic oscillator. This example is of importance for general theory, because it forms a corner-stone in the theory of radiation.

P. A. M. Dirac, *Principles of quantum mechanics*

In this chapter we introduce the concept of indefinite-frequency harmonic oscillator which is a consequence of treating oscillator’s frequency not as a classical parameter, as in the original formulation of Heisenberg, but as an eigenvalue of some operator. In the latter case a class of reducible representations of CCR algebra naturally arises. We will discuss in more detail two classes of such reducible representations involving, respectively, one oscillator and $N$ oscillators with quantized frequency.

2.1 Indefinite-frequency harmonic oscillator

Let us generalize the Hamiltonian from the previous section in a way that will make it directly applicable to the problem of field quantization in cavity quantum electrodynamics. Instead of treating frequency of a field oscillator as a classical parameter let us introduce the frequency operator [1]

$$\Omega = \sum_\omega \omega |\omega\rangle \langle \omega| \otimes 1 = \hat{\omega} \otimes 1, \quad (2.1)$$

whose eigenvalues determine the set of observable frequencies. This change of status of oscillator’s frequency, from a parameter to a quantum number, implies that its Hamiltonian becomes
\[ H = \frac{P^2}{2m} + \frac{m\Omega^2 Q^2}{2}, \]  

(2.2)

where \( Q \) and \( P \) are operators defined by

\[ Q = 1 \otimes \hat{q}, \quad P = 1 \otimes \hat{p} \]  

(2.3)

with \( \hat{q} \) and \( \hat{p} \) being the “ordinary” position and momentum operators satisfying the irreducible representation formula \([\hat{q}, \hat{p}] = i\hbar\). The associated “ordinary” annihilation and creation operators can be denoted by

\[ \hat{a}_\omega = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{q} + i\hat{p}), \quad \hat{a}^\dagger_\omega = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{q} - i\hat{p}). \]  

(2.4)

The structure of operators \( Q \) and \( P \) reflects the fact that the total Hilbert space, \( \mathcal{H} \), of a single harmonic oscillator with quantized frequency is defined as tensor product of two Hilbert spaces, the first of which is the space of frequency states, while the second one corresponds to position (momentum) states.

Annihilation and creation operators \( a_\Omega \) and \( a^\dagger_\Omega \), defined by

\[ a_\Omega = \frac{1}{\sqrt{2\hbar m\Omega}} (m\Omega \hat{Q} + iP), \quad a^\dagger_\Omega = \frac{1}{\sqrt{2\hbar m\Omega}} (m\Omega \hat{Q} - iP), \]  

(2.5)

enable us to write Hamiltonian (2.2) in several equivalent forms:

\[ H = \frac{\hbar \Omega}{2} \left( a_\Omega a^\dagger_\Omega + a^\dagger_\Omega a_\Omega \right) \]  

(2.6)

\[ = \hbar \Omega \left( a^\dagger_\Omega a_\Omega + \frac{1}{2} I \right), \]  

(2.7)

where operator \( I \) is the identity operator in \( \mathcal{H} \), \( I = 1 \otimes 1 \). Applying definitions (2.1) and (2.5) one can reexpress the Hamiltonian as

\[ H = \sum_\omega \hbar \omega |\omega\rangle \langle \omega | \otimes \left( \hat{a}^\dagger_\omega \hat{a}_\omega + \frac{1}{2} \right) \]  

(2.8)

\[ = \sum_\omega \hbar \omega \left( a^\dagger_\omega a_\omega + \frac{1}{2} I_\omega \right), \]  

(2.9)

where

\[ a_\omega = |\omega\rangle \langle \omega | \otimes \hat{a}_\omega, \]  

(2.10)

\[ a^\dagger_\omega = |\omega\rangle \langle \omega | \otimes \hat{a}^\dagger_\omega, \]  

(2.11)

\[ I_\omega = |\omega\rangle \langle \omega | \otimes 1, \]  

(2.12)
with $\hat{a}_\omega$ and $\hat{a}_\omega^\dagger$ defined by equations (2.4). Operators $a_\omega$ and $a_\omega^\dagger$, contrary to their counterparts $\hat{a}_\omega$ and $\hat{a}_\omega^\dagger$, satisfy commutation relation of the form

$$[a_\omega, a_\omega^\dagger] = I_\omega,$$  \hspace{1cm} (2.13)

with

$$\sum_\omega I_\omega = I.$$  \hspace{1cm} (2.14)

It is evident that $I_\omega$ commutes with both $a_\omega$ and $a_\omega^\dagger$, and thus relation (2.13) provides an example of a reducible representation of CCR algebra, the one we are particularly interested in. Switching from irreducible to reducible representation of CCR, determining quantum properties of the quantum field, has enormous consequences, not only mathematical, but also conceptual. One example of the latter is the need to replace the concept of definite frequency field oscillator, being a carrier of a given field mode, by its indefinite-frequency analog.

The single-oscillator Hamiltonian given by (2.9) involves, in general, infinitely many frequencies. In eigenspaces of $\Omega$, that is in subspaces of fixed frequency, $H$ acts as an ordinary Hamiltonian corresponding to an oscillator of definite frequency.

Repeating formal construction known from conventional treatments of harmonic oscillator one can define the occupation-number operator with respect to a given frequency $\omega$:

$$N_\omega = a_\omega^\dagger a_\omega = |\omega\rangle\langle \omega| \otimes \hat{a}_\omega^\dagger \hat{a}_\omega,$$  \hspace{1cm} (2.15)

that satisfies

$$[N_\omega, a_\omega] = -a_\omega, \quad [N_\omega, a_\omega^\dagger] = a_\omega^\dagger.$$  \hspace{1cm} (2.16)

The Hamiltonian now becomes

$$H = \sum_\omega \hbar \omega \left( N_\omega = \frac{1}{2} I_\omega \right).$$  \hspace{1cm} (2.17)

The term

$$\frac{1}{2} \sum_\omega \hbar \omega I_\omega = \frac{\hbar}{2} \sum_\omega \omega |\omega\rangle\langle \omega| \otimes 1 = \frac{\hbar}{2} \hat{\omega} \otimes 1 = \frac{\hbar \Omega}{2},$$  \hspace{1cm} (2.18)

plays a role of a “vacuum Hamiltonian” which, of course, is well defined even for infinitely many eigenvalues $\omega$ of the operator $\Omega$.

Assuming $|\omega, n\rangle$ is an eigenstate of $N_\omega$, with eigenvalues $n = 0, 1, 2, \ldots$, the eigenequation of the Hamiltonian can now be written as

$$H|\omega, n\rangle = \sum_\omega \hbar \omega \left( N_\omega = \frac{1}{2} I_\omega \right) |\omega, n\rangle = \hbar \omega \left( n + \frac{1}{2} \right) |\omega, n\rangle.$$  \hspace{1cm} (2.19)
The energy levels $E_{\omega,n} = \hbar \omega (n + \frac{1}{2})$ of our indefinite-frequency oscillator are thus identical to those of the usual harmonic oscillator whose frequency is $\omega$. Equation (2.16) immediately implies that states $a_\omega |\omega, n\rangle$ and $a_\omega^{\dagger} |\omega, n\rangle$ are eigenstates of $H$ with energies $(E_{\omega,n} - \hbar \omega)$ and $(E_{\omega,n} + \hbar \omega)$, respectively. The vacuum state is then $|\omega, 0\rangle$,

$$a_\omega |\omega, 0\rangle = 0,$$  \hfill (2.20)

with energy

$$E_{\omega 0} = \frac{\hbar \omega}{2}.$$  \hfill (2.21)

But this is only a part of the picture that, in fact, applies exclusively to a single eigenspace of $\Omega$. In particular, this implies that $|\omega, 0\rangle$ constitutes the vacuum state restricted to the single subspace and by no means represents the most general vacuum in the total Hilbert space $\mathcal{H}$. One therefore needs a more general definition of vacuum which would take into account the direct-sum structure of the Hilbert space. The natural choice is to define vacuum as a superposition,

$$|O\rangle = \sum_\omega O_\omega |\omega, 0\rangle,$$  \hfill (2.22)

where $O_\omega$ is a vacuum wave-function obeying the normalization condition

$$\langle O|O\rangle = \sum_\omega |O_\omega|^2 = 1.$$  \hfill (2.23)

Such a vacuum state is non-unique since, in principle, any choice of $O_\omega$ is acceptable. Notice that the vacuum subspace instead of being one-dimensional, as in the standard case, now becomes multidimensional. In the following, the vacuum state $|O\rangle$ will be called the global vacuum, while the vacua $|\omega, 0\rangle$, for arbitrary $\omega$ — the monochromatic ones.

It is straightforward to show that

$$a_\omega |O\rangle = 0,$$  \hfill (2.24)

for any $\omega$. But, on the other hand, the action of $a_\omega^{\dagger}$, which raises the number of quanta $\hbar \omega$ by one, should be treated with much care. To be more specific, the action of $a_\omega^{\dagger}$ on $|O\rangle$ yields

$$a_\omega^{\dagger} |O\rangle = O_\omega |\omega, 1\rangle,$$  \hfill (2.25)

which shows that the global vacuum state (being a superposition of monochromatic vacua) collapses to a particular one-excitation state. This effect becomes obvious in view of the fact that creation operators, typical of our reducible representation, involve projections on eigenstates of $\Omega$. As a consequence, the action of $a_{\omega'}^{\dagger}$, on $a_\omega^{\dagger} |O\rangle$, for $\omega' \neq \omega$, immediately destroys the state, i.e.

$$a_\omega^{\dagger} a_{\omega'}^{\dagger} |O\rangle = 0, \quad \text{for } \omega' \neq \omega.$$  \hfill (2.26)
This property shows that the reducible representation discussed in the present section is of limited use for the formalism of quantum optics since, for instance, creation of two different photons is impossible. This is why in the next section we analyze representations corresponding to several oscillators.

On the other hand, the number of excitations of a single mode is not limited from above even for a single oscillator, so that

\[
\frac{1}{\sqrt{n!\left|O_\omega\right|^2}}a_\omega^{\dagger n}\left|O_\right\rangle = e^{i\phi_\omega}\left|\omega, n\right\rangle,
\]

(2.27)

\[e^{i\phi_\omega} = O_\omega/\left|O_\omega\right|,\]

represents an nth excited state of an oscillator with frequency \(\omega\).

To close this section, let us consider the issue of average energy of vacuum for a single indefinite-frequency oscillator. We find

\[
\langle O|H|O \rangle = \sum_\omega \left|O_\omega\right|^2 \frac{\hbar\omega}{2}.
\]

(2.28)

The energy (2.28) is the sum of, in general, infinitely many contributions, instead of the single term \(\hbar\omega/2\) which is the case for a single oscillator of well-defined frequency \(\omega\). In fact, (2.28) is indistinguishable from the average energy obtained for an ensemble of harmonic oscillators with well defined frequencies, weighted by \(\left|O_\omega\right|^2\). Here, the presence of the sum \(\sum_\omega\) reflects the fact that the oscillator may exist simultaneously in many possible frequency states, where the probability of a particular state \(\left|\omega\right\rangle\) is given by \(\left|O_\omega\right|^2\). Since this term appears throughout this thesis many times it is convenient to use the shorthand notation

\[
Z_\omega = \left|O_\omega\right|^2.
\]

(2.29)

Probability \(Z_\omega\) enters into (2.28) as a weight for energy of the corresponding monochromatic vacuum. In the special case of the monochromatic vacuum of frequency \(\omega'\),

\[
Z_\omega = \begin{cases} 
1 & \text{for } \omega = \omega' \\
0 & \text{otherwise}
\end{cases}
\]

(2.30)
equation (2.28) reduces to the energy of a single definite-frequency oscillator, parameterized by \(\omega'\).

### 2.2 Several noninteracting bosonic indefinite-frequency oscillators

For reasons mentioned in the preceding section, we need a formalism for \(N > 1\) oscillators of indefinite frequency.

Hilbert space of \(N\) indefinite-frequency oscillators, bosonic in nature, is given by totally symmetric tensor product of \(N\) Hilbert spaces, \(\mathcal{H}\), of individual oscillators:
\[ \mathcal{H} = \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}, \]  

(2.31)

where the underline throughout this thesis serves to distinguish between entities (spaces, states, operators) defined for \( N > 1 \) and \( N = 1 \).

Assume the space \( \mathcal{H} \) of a single indefinite-frequency oscillator is spanned by vectors \( |\omega, n\rangle \). Then the basis of \( \mathcal{H} \) consists of

\[ |\omega_1, n_1\rangle \ldots |\omega_N, n_N\rangle. \]  

(2.32)

It is obvious that setting all \( n_j \) equal to zero yields the basis for the vacuum subspace of \( \mathcal{H} \).

Tensor product of one-oscillator vacua

\[ |O\rangle = |O\rangle \otimes \ldots \otimes |O\rangle \]  

(2.33)

is an example of a vacuum state for \( N \) oscillators. A specific consequence of the non-uniqueness of a single oscillator vacuum is that we may speak of separable, entangled or mixed vacuum states corresponding to \( N \geq 2 \) oscillators. So, in principle, it is perfectly legitimate to consider an entangled vacuum state of the form \( \frac{1}{\sqrt{2}} (|\omega_1, 0\rangle|\omega_1, 0\rangle + |\omega_2, 0\rangle|\omega_2, 0\rangle) \), say.

Now let \( A \) be any operator acting in \( \mathcal{H} \). The operator

\[ A^{(n)} = I^{\otimes n-1} \otimes A \otimes I^{\otimes N-n} \]  

(2.34)

acts in \( \mathcal{H} \) on states of the \( n \)th field oscillator. Since oscillators in the ensemble are bosonic by assumption, the operators (2.10)-(2.12), defined for \( N = 1 \), must be replaced by their symmetrized analogs,

\[ a_{\omega} = c_N \sum_{n=1}^{N} a_{\omega}^{(n)}, \]  

(2.35)

\[ a_{\omega}^{\dagger} = c_N^{*} \sum_{n=1}^{N} a_{\omega}^{(n)^{\dagger}}. \]  

(2.36)

Constants \( c_N \) are in principle arbitrary, but it is convenient to assign them a specific value following from some requirements. The above operators satisfy the Lie algebra

\[ [a_{\omega}, a_{\omega}^{\dagger}] = \delta_{\omega, \omega'} c_N |\omega|^{2} \sum_{n=1}^{N} I_{\omega}^{(n)} = \delta_{\omega, \omega'} L_{\omega}, \]  

(2.37)

\[ L_{\omega} = |c_N|^{2} \sum_{n=1}^{N} I_{\omega}^{(n)}. \]  

(2.38)
As mentioned earlier, for \( N = 1 \) the central elements at the right side of (2.37) satisfy the resolution of unity, \( \sum_\omega L_\omega = I \). For \( N > 1 \), on the other hand, we find

\[
\sum_\omega L_\omega = N |c_N|^2 I \otimes \ldots \otimes I = N |c_N|^2 I,
\]

where \( I \) is the identity in \( \mathcal{H} \). Requiring \( \sum_\omega L_\omega = I \) we get \( N |c_N|^2 = 1 \). In what follows we will work with \( c_N = 1/\sqrt{N} \), and thus the explicit form of multi-oscillator operators reads

\[
a_\omega = \frac{1}{\sqrt{N}} (a_\omega \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes a_\omega),
\]

\[
a_\omega^\dagger = \frac{1}{\sqrt{N}} (a_\omega^\dagger \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes a_\omega^\dagger),
\]

\[
L_\omega = \frac{1}{N} (I_\omega \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I_\omega).
\]

These operators constitute a reducible representation of CCR algebra in \( \mathcal{H} \). It is a simple task to derive some useful relations typical of this representation. For example,

\[
[a_\omega, a_\omega^\dagger^n] = nL_\omega a_\omega^\dagger^{n-1},
\]

where operator \( L_\omega \) appears instead of the unit operator, as it would be in the irreducible case. This simple example suggests that operators \( L_\omega \) will be appearing at different stages of calculations, so it makes sense to take a closer look at their properties. We address this problem in the next section.

Given average energy of a single oscillator evaluated in vacuum state \( |O\rangle \), the average energy of the collection of \( N \) such oscillators is \( N \langle O|H|O\rangle \). Although this conclusion is straightforward, it is instructive to perform a more formal reasoning. According to the discussion given below equation (2.34), the Hamiltonian for an ensemble of \( N \) noninteracting oscillators can be written as

\[
H = \sum_{n=1}^{N} H^{(n)} = H \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes H.
\]

Employing (2.9) we equivalently get

\[
H = \sum_\omega \hbar \omega \left( N_\omega + \frac{1}{2} NL_\omega \right),
\]

with \( N_\omega \) being the number of excitations operator defined as

\[
N_\omega = a_\omega^\dagger a_\omega \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes a_\omega^\dagger a_\omega,
\]
and satisfying

\begin{align}
\left[ N_\omega, a_\omega \right] &= -a_\omega, \\
\left[ N_\omega, a_\omega^\dagger \right] &= a_\omega^\dagger.
\end{align}

(2.47a)

(2.47b)

The latter two equations are simply the analogs of the relations known from a definite-frequency harmonic oscillator. Actually, including $N_\omega$, the reducible representation of CCR turns into a reducible representation of harmonic oscillator Lie algebra. At first sight it might seem more natural to define $N_\omega$ as $a_\omega^\dagger a_\omega$. The latter operator, however, does not count the number of excitations of a given frequency in an ensemble of independent oscillators (it contains extra terms and coefficients), and thus a naive translation of formulas for $N = 1$ in hope of getting those for $N > 1$ can be misleading.

Now, taking the average of (2.45) in vacuum $\left| O \right>$ of $N$ oscillators we get

\begin{equation}
\langle O\left| H \right| O \rangle = N \langle O\left| H \right| O \rangle = \sum_\omega \frac{\hbar \omega}{2} N Z_\omega,
\end{equation}

which is finite for $N < \infty$, if the regularized sum $\sum_\omega \frac{\hbar \omega}{2} Z_\omega$ is finite. In the following we will show that in order to model a quantum field by ensembles of indefinite frequency oscillators one does not need infinite values of $N$. In fact, it is sufficient to have $N < \infty$, in which case the problem of infinite vacuum energy simply does not appear.

Let us return for a moment to an irreducible representation of CCR algebra given by

\begin{equation}
\left[ \hat{a}_\omega, \hat{a}_\omega^\dagger \right] = z_\omega,
\end{equation}

(2.49)

with $z_\omega$ being a numerical function of $\omega$ (we use $z_\omega$ in order to avoid confusion with $Z_\omega$ appearing in the reducible-representation approach). In that case, the energy of vacuum becomes

\begin{equation}
\sum_\omega \frac{\hbar \omega}{2} z_\omega
\end{equation}

showing that $z_\omega$s, vanishing for $\omega \rightarrow \infty$, regularize the infinite value of vacuum energy. The reasoning is, however, unacceptable at a fundamental level, since a theory based on naive regularization (2.49) cannot be relativistically covariant [15].

### 2.3 Frequency-of-successes operator

Representation (2.40)–(2.42) has one interesting peculiarity: Central elements $I_\omega$ are the frequency-of-successes operators known from quantum laws of large numbers [17, 18, 19, 20, 21, 22, 23]. In order to develop some intuitions concerning quantum laws of large numbers let us consider the following states
\begin{align}
|\omega_1, 0\rangle |\omega_1, 0\rangle |\omega_2, 0\rangle , \tag*{(2.50)} \\
|\omega_1, 0\rangle |\omega_2, 0\rangle |\omega_1, 0\rangle , \tag*{(2.51)}
\end{align}

and
\begin{align}
|\omega_2, 0\rangle |\omega_1, 0\rangle |\omega_1, 0\rangle , \tag*{(2.52)}
\end{align}

belonging to \( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \) and representing particular vacuum states of this space. These states satisfy
\begin{align*}
I_{\omega_1} |\omega_1, 0\rangle |\omega_1, 0\rangle |\omega_2, 0\rangle & = \frac{2}{3} |\omega_1, 0\rangle |\omega_1, 0\rangle |\omega_2, 0\rangle , \\
I_{\omega_1} |\omega_1, 0\rangle |\omega_2, 0\rangle |\omega_1, 0\rangle & = \frac{2}{3} |\omega_1, 0\rangle |\omega_2, 0\rangle |\omega_1, 0\rangle , \\
I_{\omega_1} |\omega_2, 0\rangle |\omega_1, 0\rangle |\omega_1, 0\rangle & = \frac{2}{3} |\omega_2, 0\rangle |\omega_1, 0\rangle |\omega_1, 0\rangle , \\
I_{\omega_2} |\omega_1, 0\rangle |\omega_1, 0\rangle |\omega_2, 0\rangle & = \frac{1}{3} |\omega_1, 0\rangle |\omega_1, 0\rangle |\omega_2, 0\rangle , \\
I_{\omega_2} |\omega_1, 0\rangle |\omega_2, 0\rangle |\omega_1, 0\rangle & = \frac{1}{3} |\omega_1, 0\rangle |\omega_2, 0\rangle |\omega_1, 0\rangle , \\
I_{\omega_2} |\omega_2, 0\rangle |\omega_1, 0\rangle |\omega_1, 0\rangle & = \frac{1}{3} |\omega_2, 0\rangle |\omega_1, 0\rangle |\omega_1, 0\rangle .
\end{align*}

It is clear that \( I_{\omega_i} \) counts how many times \( \omega_i \) occurs in the tensor product and divides this number by \( N \), here \( N = 3 \). Physically, this means that \( I_{\omega_i} \) is an observable that represents measurements of the “yes-no” observable \( |\omega_i\rangle \langle \omega_i| \), measured on three independent oscillators. For arbitrary \( N \) the eigenvalues of \( I_{\omega_i} \) are \( \{0, 1/N, 2/N, \ldots, (N-1)/N, 1\} \).

Having identified eigenvalues and eigenstates of \( I_{\omega_i} \), it is possible to use them to spectrally decompose \( I_{\omega_i} \),
\begin{equation}
I_{\omega_i} = \sum_{s=0}^{N} \frac{S}{N} \pi_{\omega_i}(s/N). \tag{2.53}
\end{equation}

\( \pi_{\omega_i}(s/N) \) projects onto the subspace of \( \mathcal{H} \), where exactly \( s \) oscillators out of \( N \) are found in the selected mode \( \omega \). As we will see, the projectors \( \pi_{\omega_i}(s/N) \) play a crucial role in quantum optical applications involving fields quantized in reducible \( N \)-oscillator representations of CCR algebra. Projector \( \pi_{\omega_i}(s/N) \) can be written as
\begin{equation}
\pi_{\omega_i}(s/N) = \sum_{\lambda_1 + \ldots + \lambda_N = s} P^{(\lambda_1)}_{\omega} \otimes \ldots \otimes P^{(\lambda_N)}_{\omega}, \quad \lambda_i \in \{0,1\}, \tag{2.54}
\end{equation}

with \( P^{(1)}_{\omega} \) and \( P^{(0)}_{\omega} \) being binary operators defined on single-oscillator Hilbert space \( \mathcal{H} \), and providing information whether an oscillator is in mode \( \omega \) in question. Formally, we have
\[ P_{\omega}^{(1)} = |\omega\rangle \langle \omega | \otimes 1, \quad (2.55) \]
\[ P_{\omega}^{(0)} = \sum_{\omega' \neq \omega} |\omega'\rangle \langle \omega' | \otimes 1 = I - P_{\omega}^{(1)}. \quad (2.56) \]

The right hand side of (2.54) consists of a certain number of terms each of which is a tensor product of exactly \( s \) projectors \( P_{\omega}^{(1)} \) and \( N - s \) projectors \( P_{\omega}^{(0)} \). To specify the precise number of these terms it is sufficient to realize that it corresponds to the number of possible combinations of distributing \( s \) ones and \( N - s \) zeros between \( N \) cells. Equation (2.54) can, therefore, be written as

\[ \pi_{\omega}(s/N) = \sum_{\text{permutations}} P_{\omega}^{(1)} \otimes \ldots \otimes P_{\omega}^{(1)} \otimes P_{\omega}^{(0)} \otimes \ldots \otimes P_{\omega}^{(0)}, \quad (2.57) \]

where the sum runs over all \( \binom{N}{s} \) possible ways of distributing \( s \) projectors \( P_{\omega}^{(1)} \) over \( N \) sites. This equation shows that \( \pi_{\omega}(s/N) \) is a symmetrized sum of \( \binom{N}{s} \) projectors each of which is composed of \( N \) projectors that act in subspaces of individual oscillators. Physically, \( \pi_{\omega}(s/N) \) projects onto a subspace of states corresponding to exactly \( s \) oscillators prepared in a selected mode \( \omega \). Clearly, taking products of projectors \( \pi_{\omega}(s/N) \) with different \( s \) and \( \omega \) we obtain projectors onto subspaces of states with given properties specified by the number of oscillators in definite modes.

Applying explicitly the definition (2.54) it can be shown that for arbitrary but fixed \( \omega \)

\[ \pi_{\omega}(s_1/N)\pi_{\omega}(s_2/N) = \delta_{s_1,s_2}\pi_{\omega}(s_1/N), \quad (2.58) \]

meaning that subspaces corresponding to different numbers of oscillators that exist in the selected mode are mutually orthogonal.

On the other hand, it can be verified that for \( \omega_1 \neq \omega_2 \) and \( s_1 + s_2 > N \)

\[ \pi_{\omega_1}(s_1/N)\pi_{\omega_2}(s_2/N) = 0. \quad (2.59) \]

A good way to understand this is to realize that the first projector, \( \pi_{\omega_1}(s_1/N) \), projects on states where exactly \( s_1 \) oscillators are in mode \( |\omega_1\rangle \), whereas the remaining \( N - s_1 \) oscillators are in frequency states orthogonal to \( |\omega_1\rangle \). Projector \( \pi_{\omega_2}(s_2/N) \) involves tensor products of \( s_2 \) projectors on \( |\omega_2\rangle \), where \( \langle \omega_2|\omega_1\rangle = 0 \). Since \( s_2 > N - s_1 \), each term of (2.59) must involve at least one projection of \( |\omega_1\rangle \) on the orthogonal direction \( |\omega_2\rangle \), hence (2.59) vanishes.

For \( s_1 = 0 \) and \( s_2 = N \) we analogously prove, for all \( \omega_1 \neq \omega_2 \),

\[ \pi_{\omega_1}(0/N)\pi_{\omega_2}(N/N) = \pi_{\omega_2}(N/N). \quad (2.60) \]

Indeed, \( \pi_{\omega_2}(N/N) \) projects on states with all oscillators in \( |\omega_2\rangle \). All such states belong to the subspace defined by \( \pi_{\omega_1}(0/N) \), i.e. the one with no oscillator in state \( |\omega_1\rangle \).
Another key property of operator \( \pi_\omega(s/N) \), valid for any \( \omega \), is the resolution of unity in \( \mathcal{H} \):

\[
I = \sum_{s=0}^{N} \pi_\omega(s/N). \tag{2.61}
\]

To prove this relation it suffices to write 1-oscillator identity operator as \( I = P^{(1)}_\omega + P^{(0)}_\omega \) and apply definition (2.57):

\[
I = (P^{(1)}_\omega + P^{(0)}_\omega) \otimes \ldots \otimes (P^{(1)}_\omega + P^{(0)}_\omega)
= \pi_\omega(N/N) + \pi_\omega(0/N) + \sum_{s=1}^{N-1} \pi_\omega(s/N)
= \sum_{s=0}^{N} \pi_\omega(s/N). \tag{2.62}
\]

As has been mentioned above, \( \omega \) is arbitrary but fixed, and thus this identity resolution can be written in various equivalent ways, using different \( \omega \)s. In general, we can therefore write

\[
I = \sum_{s_1=0}^{N} \pi_{\omega_1}(s_1/N) \ldots \sum_{s_L=0}^{N} \pi_{\omega_L}(s_L/N) \tag{2.63}
\]

or, making use of (2.59),

\[
I = \sum_{s_1+\ldots+s_L \leq N} \pi_{\omega_1}(s_1/N) \ldots \pi_{\omega_L}(s_L/N). \tag{2.64}
\]

As an example let us take the case of \( N = 2 \) oscillators and two modes \( \omega_1 \) and \( \omega_2 \). We find

\[
I = \pi_{\omega_1}(0/2)\pi_{\omega_2}(0/2) + \pi_{\omega_2}(2/2) + \pi_{\omega_1}(2/2)
+ \pi_{\omega_1}(0/2)\pi_{\omega_2}(1/2) + \pi_{\omega_1}(1/2)\pi_{\omega_2}(0/2) + \pi_{\omega_1}(1/2)\pi_{\omega_2}(1/2). \tag{2.65}
\]

With the above identity resolution defined on \( \mathcal{H} \) any arbitrary state \( |\psi\rangle \) belonging to \( \mathcal{H} \) can be written as

\[
|\psi\rangle = \sum_{s_1+\ldots+s_L \leq N} |\psi\rangle_{(s_1,\omega_1),\ldots,(s_L,\omega_L)}, \tag{2.66}
\]

where

\[
|\psi\rangle_{(s_1,\omega_1),\ldots,(s_L,\omega_L)} \in \pi_{\omega_1}(s_1/N) \ldots \pi_{\omega_L}(s_L/N)\mathcal{H}. \tag{2.67}
\]

The condition \( s_1+\ldots+s_L \leq N \) imposes certain restrictions on practical use of this resolution of identity: The number of orthogonal subspaces (being eigenspaces of \( L_\omega \)) that decompose
cannot exceed $N$. Another consequence is that each state is indexed by a finite sequence of modes $\omega_1, \ldots, \omega_L$, $L \leq N$, even if the Hilbert space spanned by $|\omega\rangle$ is infinite-dimensional. The extreme case of $s_i = 1$ for $i = 1, 2, \ldots, N$ corresponds to a situation where we are interested in $N$ different frequencies distributed among $N$ oscillators.

Technically, this result is very important as it gives us the freedom of applying the identity resolution in forms tailored to concrete physical situations. As an illustration let us take the Jaynes-Cummings model of the atom-field interaction, where the interaction happens through a single mode $\omega$ only, and thus the most convenient identity resolution of form (2.62) is

$$|\psi\rangle = \sum_{s=0}^{N} \pi_{\omega}(s/N)|\psi\rangle.$$  \hspace{2cm} (2.68)

If infinitely many modes are involved, the above resolutions of identity allow us to split the Hilbert space into subspaces involving at most $N$ different modes (this fact may be of great importance in the context of resonance fluorescence).

The last point to be mentioned is the distinction between $s_1 + \ldots + s_L = N$ and $s_1 + \ldots + s_L < N$. The former imposes a restriction by specifying a closed list of frequencies which may be represented in an ensemble of oscillators: $s_1$ oscillators of frequency $\omega_1$, $s_2$ oscillators of frequency $\omega_2$, and so on until $s_L$ oscillators of frequency $\omega_L$. The two extreme cases are $s_1 = N$, and $s_1 + \ldots + s_N = N$, where each $s_i = 1$.

The second case mentioned above is $s_1 + \ldots + s_L < N$, where the list of ensemble’s frequencies is left open.

These results show that the total Hilbert space can be partitioned into orthogonal subspaces, where the exact way of partitioning is dictated by the physical background of the problem.

### 2.4 $N$-oscillator analogs of $n$-photon states

$N$-oscillator vacuum states space is spanned by $|\omega_1, 0\rangle \ldots |\omega_N, 0\rangle$. In this thesis we assume that vacua are given by pure product states

$$|O\rangle = |O\rangle \otimes \ldots \otimes |O\rangle = \sum_{\omega_1, \ldots, \omega_N} O_{\omega_1} \ldots O_{\omega_N} |\omega_1, 0\rangle \ldots |\omega_N, 0\rangle,$$  \hspace{2cm} (2.69)

where the $O_{\omega}$s play a role of a single oscillator wave-function, normalized by

$$\sum_{\omega} |O_{\omega}|^2 = 1.$$  \hspace{2cm} (2.70)

Recall that $Z_\omega = |O_{\omega}|^2$ is the probability that a given oscillator has frequency $\omega$.

Let us now consider the analog of an ordinary 1-photon state, that is $O_{\omega}^\dagger |O\rangle$. Its norm is
\[ \langle O | a_\omega a_\omega^\dagger | O \rangle = \langle O | I_\omega | O \rangle \]
\[ = \frac{1}{N} \langle O | \cdots \langle O | (I_\omega \otimes \cdots \otimes I + \cdots \otimes I_\omega) | O \rangle \cdots | O \rangle \]
\[ = \langle O | I_\omega | O \rangle \]
\[ = |O_\omega|^2 = Z_\omega. \] (2.71)

The same result could be obtained by making use of the spectral decomposition of \( I_\omega \), given by (2.53). Indeed, we get
\[ \langle O | a_\omega a_\omega^\dagger | O \rangle = \sum_{s=0}^{N} s \frac{s}{N} \langle O | \pi_\omega \left( \frac{s}{N} \right) | O \rangle \] (2.72)
\[ = \sum_{s=0}^{N} s \frac{s}{N} \sum_{\lambda_1 + \cdots + \lambda_N = s} \langle O | P_\omega^{(\lambda_1)} \otimes \cdots \otimes P_\omega^{(\lambda_N)} | O \rangle. \] (2.73)

Expectation value of projector \( P_\omega^{(\lambda_1)} \otimes \cdots \otimes P_\omega^{(\lambda_N)} \), with any sequence of \( \lambda_i \)s, \( \lambda_i \in \{0, 1\} \), satisfying \( \lambda_1 + \cdots + \lambda_N = s \), is given by
\[ \langle O | P_\omega^{(\lambda_1)} \otimes \cdots \otimes P_\omega^{(\lambda_N)} | O \rangle = \langle O | P_\omega^{(1)} | O \rangle^s \langle O | P_\omega^{(0)} | O \rangle^{N-s} = Z_\omega^s (1 - Z_\omega)^{N-s}, \] (2.74)

since each expectation value of \( P_\omega^{(1)} \) or \( P_\omega^{(0)} \) taken in \( |O\rangle \) can be simply replaced by \( Z_\omega \) or \( 1 - Z_\omega \), respectively. Summation over \( \lambda_1 + \cdots + \lambda_N = s \) in (2.73) means summation over all the possible arrangements of \( s \) ones and \( N - s \) zeros distributed among \( N \) places in irrelevant order, which gives \( \binom{N}{s} = \frac{N!}{s!(N-s)!} \) different arrangements. Thus the expectation value of \( \pi_\omega \left( \frac{s}{N} \right) \) in \( |O\rangle \) can be written as
\[ \langle O | \pi_\omega \left( \frac{s}{N} \right) | O \rangle = \sum_{\lambda_1 + \cdots + \lambda_N = s} \langle O | P_\omega^{(\lambda_1)} \otimes \cdots \otimes P_\omega^{(\lambda_N)} | O \rangle \]
\[ = \binom{N}{s} Z_\omega^s (1 - Z_\omega)^{N-s} \]
\[ = \frac{N!}{(N-s)!s!} Z_\omega^s (1 - Z_\omega)^{N-s}. \] (2.75)

The right side is the binomial distribution
\[ b(s; Z_\omega, N) := \frac{N!}{(N-s)!s!} Z_\omega^s (1 - Z_\omega)^{N-s}, \] (2.76)
describing the probability that \( N \) Bernoulli trials (field oscillators) result in exactly \( s \) successes (oscillator in mode \( \omega \)) and \( N - s \) failures (oscillator in mode non-\( \omega \)). Applying these
remarks to (2.72) and (2.73), and making use of the well-known property of the binomial
distribution, we can finally write

$$\langle O|a_\omega a_\omega^\dagger |O\rangle = \sum_{s=0}^{N} \frac{s}{N} b(s; Z_\omega, N) = Z_\omega,$$

(2.77)

which agrees with (2.71) obtained by simpler arguments.

The Bernoulli scheme is appropriate here, since:

1. By assumption the oscillators (trials) are independent.

2. States of each oscillator belong to one of the two classes — in the selected frequency
(state $|\omega\rangle$ (success), or in $|\omega'\rangle$, orthogonal to $|\omega\rangle$ (failure). These two classes correspond
to the possible outcomes of a trial.

3. Probabilities of successes and failures remain the same for each trial, and are equal to
$Z_\omega$ and $1 - Z_\omega$, respectively.

Taking this trial-oscillator identification into account, $b(s; Z_\omega, N)$ corresponds to the prob-
ability of $s$ successes in $N$ trials or, in physical terms, to the probability of the event that
$s$ out of $N$ oscillators are found in mode $\omega$. As the oscillators are independent, $s$ is a dis-
crete random variable taking values $0, 1, \ldots, N$, and $b(s; Z_\omega, N)$ is the distribution with $NZ_\omega$
being the most probable number of oscillators in mode $\omega$.

These calculations can be easily generalized to $n$-photon states, provided all photons are
of the same frequency. In that case we get

$$\langle O|a_\omega^n a_\omega^\dagger^n |O\rangle = n! \sum_{s=0}^{N} \left( \frac{s}{N} \right)^n \langle O|\pi_{\omega}(s/N)|O\rangle
= n! \langle O|L_\omega^n|O\rangle.$$

(2.78)

Employing spectral decomposition (2.53) and the property given by (2.58), this equation
can be rewritten as follows:

$$\langle O|a_\omega^n a_\omega^\dagger^n |O\rangle = n! \sum_{s=0}^{N} \left( \frac{s}{N} \right)^n \langle O|\pi_{\omega}(s/N)|O\rangle
= n! \sum_{s=0}^{N} \left( \frac{s}{N} \right)^n b(s; Z_\omega, N),$$

(2.79)

representing an average of the discrete random variable $n!(s/N)^n$. Normalized analogs of
ordinary $n$-photon states are therefore

$$\frac{1}{\sqrt{n!}} \frac{1}{\sqrt{\sum_{s=0}^{N}(s/N)^n b(s; Z_\omega, N)}} a_\omega^n |O\rangle.$$

(2.80)
Now, it is important to realize that state (2.80) corresponds to \( n \) excitations (photons) of energy \( E = \hbar \omega \) distributed in the \( N \)-oscillator bosonic gas in all the possible ways. Thus, it is a superposition of states

\[
|\omega, n\rangle|\omega_2, 0\rangle \ldots |\omega_N, 0\rangle,
\]

\[
|\omega, n-1\rangle|\omega, 1\rangle|\omega_2, 0\rangle \ldots |\omega_N, 0\rangle,
\]

\[
|\omega, n-1\rangle|\omega_1, 0\rangle|\omega, 1\rangle \ldots |\omega_N, 0\rangle,
\]

and so on. Notice, however, that the term “\( n \)-photon state” should, in principle, be used to describe any state spanned by the basis vectors given above and, therefore, not all analogs of \( n \)-photon states can be generated from the vacuum \(|\Omega\rangle\) by repeated action of the creation operator. A simple example of a state that involves \( n \) excitations but is not of form (2.80) is

\[
f(L_\omega)a_\omega^\dagger n|\Omega\rangle,
\]

where \( f(L_\omega) \) is any function of operator \( L_\omega \).

What is characteristic of states analyzed up to this point is that they corresponded to excitations of a single frequency (mode). Now, let us consider non-monochromatic excited states of an ensemble of \( N \) oscillators.

The first obvious observation is that such an ensemble is able to store photons of at most \( N \) different frequencies. An attempt to excite more than \( N \) modes inevitably ends in destruction of a state, which is analogous to what we have discussed at equation (2.26). As an example let us consider \( a_\omega^\dagger \ldots a_\omega^\dagger|\Omega\rangle \), where all the modes are different and \( L \leq N \). The normalization condition leads to

\[
\langle O|a_\omega^\dagger \ldots a_\omega^\dagger a_\omega^\dagger \ldots a_\omega^\dagger|O\rangle = \langle O|L_\omega^\dagger \ldots L_\omega^\dagger|O\rangle.
\]

Further, making use of the spectral decomposition we get

\[
\langle O|L_\omega^\dagger \ldots L_\omega^\dagger|O\rangle = \sum_{s_1=0}^{N} \cdots \sum_{s_L=0}^{N} \frac{s_1}{N} \cdots \frac{s_L}{N} \langle O|\pi_\omega_1(s_1/N) \ldots \pi_\omega_L(s_L/N)|O\rangle,
\]

where terms of the form

\[
\langle O|\pi_\omega_1(s_1/N) \ldots \pi_\omega_L(s_L/N)|O\rangle
\]

represent probabilities of finding, in \( N \) trials, \( s_1 \) times color \( \omega_1 \), \( \ldots \), \( s_L \) times color \( \omega_L \), and \( N - s_1 - \cdots - s_L \) times a color different from any out of list \( \omega_1, \ldots, \omega_L \). Formally, events of this kind are described by multinomial distributions, and thus
\[ \langle O | \pi \omega_1 (s_1/N) \ldots \pi \omega_L (s_L/N) | O \rangle = \frac{N!}{s_1! \ldots s_L!(N - s_1 - \ldots - s_L)!} Z_{\omega_1}^{s_1} \ldots Z_{\omega_L}^{s_L} (1 - Z_{\omega_1} - \ldots - Z_{\omega_L})^{N - s_1 - \ldots - s_L}. \]  

(2.84)

If \( f_1, \ldots, f_L : [0, 1] \to \mathbb{R} \) are continuous functions, then

\[ \langle O | f_1(I_{\omega_1}) \ldots f_L(I_{\omega_L}) | O \rangle = \sum_{s_1 \ldots s_L} f_1(s_1/N) \ldots f_L(s_L/N) \langle O | \pi \omega_1 (s_1/N) \ldots \pi \omega_L (s_L/N) | O \rangle. \]  

(2.85)

The weak law of large numbers (Feller’s theorem \([24, 25]\)) implies then that

\[ \lim_{N \to \infty} \langle O | f_1(I_{\omega_1}) \ldots f_L(I_{\omega_L}) | O \rangle = f_1(Z_{\omega_1}) \ldots f_L(Z_{\omega_L}). \]  

(2.86)

We will later see that this property of the reducible \( N \)-oscillator representation will play a role of a correspondence principle mapping quantum optics based on reducible representations into \emph{automatically regularized} forms of standard quantum optics, based on irreducible representations. The regularizations will follow from the fact that probabilities \( Z_{\omega} = |O_{\omega}|^2 \) tend to 0 as \( \omega \) tends to infinity, and no \emph{ad hoc} regularization is needed.

To conclude this section, let \( |\omega, n\rangle \) be the basis of \( \mathcal{H} \), the Hilbert space of a single indefinite-frequency harmonic oscillator. Let us define the projector

\[ \Pi(n) = \sum_{\omega} |\omega, n\rangle \langle \omega, n| = 1 \otimes |n\rangle \langle n|, \]  

(2.87)

that determines the probability of finding an \( n \)-th excited state of the oscillator. Projector

\[ \Pi(n) = \sum_{n_1 + \ldots + n_N = n} \Pi(n_1) \otimes \ldots \otimes \Pi(n_N) \]  

(2.88)

corresponds then to the probability of finding \( n \) excitations distributed over all the \( N \) oscillators irrespective of their frequency. Now let us consider a normalized state

\[ |\psi\rangle = C (\sum_{\omega} c_{\omega} a_{\omega}^\dagger)^m |O\rangle \]  

(2.89)

where \( C \) is a normalization factor. Then,

\[ \langle \psi | \Pi(n) | \psi \rangle = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \]  

(2.90)

This formula clearly explains why it is justified to speak of analogs of \( n \)-photon states.
2.5 $N$-oscillator analogs of coherent states

Quantum mechanics textbooks give two equivalent definitions of coherent states of a harmonic oscillator [26, 27, 28]. First, these are the states generated by displacement operators acting on the oscillator ground state. Second, the eigenstates of the boson annihilation operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. In this section we propose a generalization of the coherent states to the quantum system of $N$ indefinite frequency oscillators, where the two definitions given above are not exactly equivalent.

First, let us define the displacement operator as

$$D(\{\alpha\}) = e^{\sum_\omega (\alpha_\omega \hat{a}_\omega^\dagger - \alpha^*_\omega \hat{a}_\omega)} = e^{-\frac{1}{2} \sum_\omega |\alpha_\omega|^2} e^{\sum_\omega \alpha_\omega \hat{a}_\omega^\dagger} e^{-\sum_\omega \alpha^*_\omega \hat{a}_\omega}. \quad (2.91)$$

It is straightforward to show that it is unitary,

$$D(\{\alpha\})^\dagger = D(-\{\alpha\}) = D^{-1}(\{\alpha\}), \quad (2.92)$$

and acts as a shift operator

$$D^\dagger(\{\alpha\}) a_\omega D(\{\alpha\}) = a_\omega + \alpha_\omega I_\omega, \quad (2.93)$$

$$D^\dagger(\{\alpha\}) a_\omega^\dagger D(\{\alpha\}) = a_\omega^\dagger + \alpha^*_\omega I_\omega. \quad (2.94)$$

The latter two equations imply

$$a_\omega D(\{\alpha\}) |O\rangle = \alpha_\omega L_\omega D(\{\alpha\}) |O\rangle, \quad (2.95)$$

which is a generalized eigenvalue equation. Acting on both sides of (2.95) with spectral projector $\pi_\omega(s/N)$,

$$\pi_\omega(s/N) a_\omega D(\{\alpha\}) |O\rangle = \pi_\omega(s/N) \alpha_\omega I_\omega D(\{\alpha\}) |O\rangle = \alpha_\omega \pi_\omega(s/N) L_\omega D(\{\alpha\}) |O\rangle, \quad (2.96)$$

we obtain an analog of the usual coherent-state eigenvalue equation restricted to the eigenspace of $L_\omega$ with eigenvalue $s/N$,

$$\pi_\omega(s/N) a_\omega D(\{\alpha\}) |O\rangle = \pi_\omega(s/N) \alpha_\omega \frac{s}{N} D(\{\alpha\}) |O\rangle. \quad (2.97)$$

Notice that the right-hand side contains $\alpha_\omega$ rescaled by the fraction $s/N$. The usual eigenequation for a displacement operator,

$$\hat{a} D(\{\alpha\}) |0\rangle = \alpha D(\{\alpha\}) |0\rangle, \quad (2.98)$$
is a bit misleading, since it is satisfied for the particular irreducible representation $[\hat{a}, \hat{a}^\dagger] = 1$. It suffices to take any other irreducible representation, say $[\hat{a}, \hat{a}^\dagger] = Z$, with $Z = \text{const}$, to get

$$\hat{a}D(\{\alpha\})|0\rangle = \alpha Z D(\{\alpha\})|0\rangle,$$  

(2.99)

with parameter $\alpha$ rescaled by constant $Z$. This simple example allows us to believe that the eigenspaces of $I_\omega$ are the spaces where irreducible representations of the CCR algebra act. Indeed, the central elements $I_\omega$ are proportional to identity operators of the relevant spaces. In other words, spectral projectors $\pi_\omega(s/N)$ allow us to split the $N$-indefinite-frequency-oscillator representation into a direct sum of irreducible representations of CCR. Let us define

$$a_\omega(s) = \pi_\omega(s/N) a_\omega(s/N) = a_\omega(s/N) a_\omega,$$  

(2.100)

$$a_\omega(s)^\dagger = \pi_\omega(s/N) a_\omega^\dagger(s/N) = a_\omega^\dagger(s/N) a_\omega.$$  

(2.101)

Then, the corresponding representation of CCR is

$$[a_\omega(s), a_\omega(s)^\dagger] = \frac{s}{N} \pi_\omega(s/N).$$  

(2.102)

This means, that in subspace $\mathcal{H}_\omega(s) = \pi_\omega(s/N)\mathcal{H}$ the representation is simply

$$[a_\omega(s), a_\omega(s)^\dagger] = \frac{s}{N}.$$  

(2.103)

For $s = 0$ we find $[a_\omega(0), a_\omega(0)^\dagger] = 0$. For $s \geq 1$ we can divide both sides of (2.103) by $s/N$, and define $\tilde{a}_\omega(s) = a_\omega(s)/\sqrt{s/N}$. Then, finally, in $\mathcal{H}_\omega(s)$ the representation becomes just

$$[\tilde{a}_\omega(s), \tilde{a}_\omega(s)^\dagger] = 1.$$  

(2.104)

This trick allows us to derive various properties of coherent states by means of the standard techniques known from irreducible representations. Generalizing this observation one concludes that any problem involving the indefinite-frequency reducible representation can be split into a direct sum of problems that can be treated by the usual techniques. A practical consequence is that any problem that can be solved algebraically in standard quantum optics, can be analogously solved in a theory based on reducible representations.

### 2.6 Statistics of excitations

Equipped with these facts we are now in position to analyze the statistics of coherent states. First, we calculate the probability of finding exactly $n$ excitations in a coherent state $|\alpha\rangle = D(\{\alpha\})|\Omega\rangle$. Employing projector $\Pi(n)$ given by (2.88), we get
The limit $N \to \infty$ is equivalent to $q \to 1$, which is known in information theory as the Shannon limit of Rényi $q$-entropies. The expression

$$
\left( \sum_\omega Z_\omega (e^{\lambda|\alpha_\omega|^2})^{1-q} \right)^{\frac{1}{1-q}}
$$

is Kolmogorov-Nagumo average of random variable $e^{\lambda|\alpha_\omega|^2}$ [29].
Ensembles of $N$ indefinite-frequency coherent-state oscillators satisfy Rényi’s $q$-statistics with $q = 1 - 1/N$. The limiting case $q \to 1$,

\[
\lim_{q \to 1} \left( \sum_{\omega} Z_{\omega} e^{\lambda(1-q)|\alpha_\omega|^2} \right)^{\frac{1}{1-q}} = e^{\lambda \sum_{\omega} |\alpha_\omega|^2},
\]

(2.110)

reproduces the Poisson distribution typical of standard coherent states, but with one very important and subtle difference: Standard coherent states would imply $e^{\lambda \sum_{\omega} |\alpha_\omega|^2}$ and not $e^{\lambda \sum_{\omega} |\alpha_\omega|^2}$. The limiting case of the reducible representation is thus not exactly the irreducible one, but its automatically regularized version. This fact will have fundamental consequences for quantum field theory based on reducible $N$-indefinite-oscillator representations of CCR.

The second problem we address here is to calculate the average number of photons of a given color (frequency) taken in a monochromatic coherent state $|\alpha_\omega\rangle$. The average of $N_\omega$ taken in $|\alpha_\omega\rangle$ can be written as

\[
\langle N_\omega \rangle = \langle D|D\rangle\langle \alpha_\omega|N_\omega D(\alpha_\omega)|O\rangle = \langle D|e^{-|\alpha_\omega|^2L_\omega}\alpha_\omega a_\omega^\dagger e^{\alpha_\omega a_\omega^\dagger}N_\omega e^{\alpha_\omega a_\omega^\dagger}e^{-\alpha_\omega a_\omega^\dagger}|O\rangle
\]

\[
= \langle D|e^{-|\alpha_\omega|^2L_\omega}\alpha_\omega a_\omega^\dagger N_\omega e^{\alpha_\omega a_\omega^\dagger}|O\rangle
\]

\[
= \sum_{n,n'=0}^{\infty} \frac{(\alpha_\omega^*)^{n'}}{n!} \frac{\alpha_\omega^n}{n'} \langle D|e^{-|\alpha_\omega|^2L_\omega}\alpha_\omega^\dagger a_\omega^\dagger n|O\rangle
\]

\[
= \sum_{n=0}^{\infty} n \frac{|\alpha_\omega|^{2n}}{n!} \langle D|e^{-|\alpha_\omega|^2L_\omega}\alpha_\omega^\dagger a_\omega^\dagger n|O\rangle
\]

\[
= \sum_{n=0}^{\infty} n \frac{|\alpha_\omega|^{2n}}{n!} \langle D|e^{-|\alpha_\omega|^2L_\omega}F_\omega^n|O\rangle
\]

\[
= \sum_{n=0}^{\infty} n \frac{|\alpha_\omega|^{2n}}{n!} \sum_{s=0}^{N} e^{-|\alpha_\omega|^2 s/N} \left( \frac{s}{N} \right)^n \langle D|\pi_\omega(s/N)|O\rangle
\]

\[
= |\alpha_\omega|^2 \sum_{s=0}^{N} \langle D|s/N\pi_\omega(s/N)|O\rangle
\]

\[
= |\alpha_\omega|^2 \langle D|O\rangle
\]

\[
= |\alpha_\omega|^2 Z_\omega,
\]

(2.111)

which is the standard solution multiplied by the probability of finding a single oscillator in mode $\omega$. Note that although photon statistics depends here on the number of oscillators $N$, the average number of photons remains independent of $N$. 29
Chapter 3

Jaynes-Cummings model in lossless cavities. Irreducible vs. reducible approach

Now, what should we do in the present situation? (...) There are enormous possibilities, because this theory of Heisenberg is very powerful, much more powerful than classical mechanics. It is powerful because the dynamical variables that can occur in it can be of a very general nature.

P. A. M. Dirac to a gathering of Nobel Laureates, Lindau, July 1, 1982

The simplest model providing quantum mechanical description of light-matter interaction is the Jaynes-Cummings model introduced by E. T. Jaynes and F. W. Cummings in 1963 as an idealization of the matter-field coupling in free space [8]. The model describes a typical situation encountered in contemporary experiments in cavity quantum electrodynamics domain, when a single two-level atom is coupled to a single cavity mode. Assuming that each mode of a quantum field is represented by a single harmonic oscillator with corresponding frequency, the Jaynes-Cummings model describes coupling of two objects: a single two-level atom and a single harmonic oscillator. Peculiarities of the model and its particular simplicity and solvability come directly from reduction of the number of modes through which the atom and the field can interact. Technically, such an assumption drastically simplifies calculations, since all but one modes are ignored in evolution of the system. We briefly discuss these problems in section 3.1. Things, however, get complicated when we come to reducible representations, where the one-to-one correspondence between cavity modes and harmonic oscillators is no longer valid. We tackle this problem in section 3.2.

3.1 Standard approach

The Hamiltonian that describes a system composed of a two-level atom and a single field-oscillator of well-defined frequency $\omega$ can be written as
\[ H = \hbar \Omega = \hbar (\Omega_A + \Omega_F + \Omega_{\text{int}}), \]  
where the meaning of \( \Omega \) is different from that used in the preceding chapters. Operators \( \Omega_A \) and \( \Omega_F \) are, respectively, the free atom and the free field Hamiltonians defined as

\[ \begin{align*}
\Omega_A &= \omega_0 R_3, \\
\Omega_F &= \omega \hat{a}_\omega \hat{a}_\omega^\dagger
\end{align*} \]

and \( \Omega_{\text{int}} \) is the rotating-wave approximation (RWA) term

\[ \Omega_{\text{int}} = g R_+ a_\omega + g R_- a_\omega^\dagger. \]

Parameter \( g \) describes strength of the atom-mode coupling (for simplicity it is assumed to be real), operators \( R_+ = |e\rangle\langle g|, R_- = |g\rangle\langle e| \) are the so-called atomic transition operators and \( R_3 = \frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \) is the atomic inversion operator. These operators satisfy the algebra

\[ \begin{align*}
[R_+, R_-] &= 2R_3, \\
[R_3, R_\pm] &= \pm R_\pm.
\end{align*} \]

In the above equations \( |g\rangle \) and \( |e\rangle \) represent, respectively, the lower and the upper state of the atom confined in a cavity, \( \omega_0 \) is the atomic transition frequency, and \( \omega \) the frequency of the field (harmonic oscillator) in the cavity.

First observation is that states \( |g, n\rangle \) and \( |e, n-1\rangle \), \( n = 1, 2, \ldots \), are eigenstates of \( \Omega_A + \Omega_F \) with the corresponding eigenvalues \( -\omega_0/2 + n\omega \) and \( \omega_0/2 + (n-1)\omega \), respectively. The special state \( |g, 0\rangle \) is the lowest-energy state of the system.

Both \( |g, n\rangle \) and \( |e, n-1\rangle \) store \( n \) excitations: either as \( n \) field-quanta, as in the case of \( |g, n\rangle \), or as the sum of \( n-1 \) field quanta and single atomic excitation, as in \( |e, n-1\rangle \) state. For \( \Delta = \omega_0 - \omega \) equal or close to zero these uncoupled atom-cavity energy states are degenerate or nearly degenerate. Pairs of these states for successive values of the total number of excitations \( n = 1, 2, \ldots \) form doublets of states whose energies are separated by \( \omega \). The special state \( |g, 0\rangle \) is the lowest state of the system and remains uncoupled from all the other states within RWA approximation and under the assumption of ideal (lossless) cavities. This state is therefore not particularly interesting when considering ideal atom-field interaction, as the system is unable to evolve nontrivially starting from it.

It is obvious that the interaction part of the Hamiltonian, \( \Omega_{\text{int}} \), connects states that belong to a given doublet only, while its matrix elements for states from distinct doublets equal zero. Therefore, the Hamiltonian of the total system has a block-diagonal form and the problem of its diagonalization can be divided into diagonalization of each block separately. Note also that each block can be characterized by a definite number of excitations, \( n \). In effect, the total Hamiltonian restricted to a given single doublet composed of the two so-called bare states \( \{ |g, n\rangle, |e, n-1\rangle \} \) has the form
\[ H = \hbar \Omega = \hbar \left( \frac{\omega_0}{2} + (n - 1)\omega \frac{g\sqrt{n}}{g\sqrt{n}} - \frac{\omega_0}{2} + n\omega \right). \] (3.7)

The diagonal terms of (3.7) correspond to the free part of the Hamiltonian, while the off-diagonal terms correspond to the interaction term. The eigenvalues of \( \Omega \) are:

\[ \Omega_\pm = (n - \frac{1}{2})\omega \pm \frac{1}{2}\omega_R, \] (3.8)

where

\[ \omega_R = \sqrt{\Delta^2 + 4g^2n} \] (3.9)

is the so-called Rabi frequency.

Frequencies \( \Omega_+ \) and \( \Omega_- \) are separated by \( \omega_R \), which is non-zero even for strict resonance. The corresponding normalized eigenstates of \( \Omega \) are linear combinations of the bare states and are referred to as the dressed states,

\[ |\Omega_+\rangle = \left( \begin{array}{c} \sqrt{\frac{\omega_R + \Delta}{2\omega_R}} \\ \sqrt{\frac{\omega_R - \Delta}{2\omega_R}} \end{array} \right) = \cos \theta |e, n - 1\rangle + \sin \theta |g, n\rangle, \] (3.10a)

\[ |\Omega_-\rangle = \left( \begin{array}{c} -\sqrt{\frac{\omega_R - \Delta}{2\omega_R}} \\ \sqrt{\frac{\omega_R + \Delta}{2\omega_R}} \end{array} \right) = -\sin \theta |e, n - 1\rangle + \cos \theta |g, n\rangle, \] (3.10b)

where

\[ \cos \theta = \sqrt{\frac{\omega_R + \Delta}{2\omega_R}}, \quad \sin \theta = \sqrt{\frac{\omega_R - \Delta}{2\omega_R}}. \] (3.11)

It is obvious that taking \( \Delta = 0 \) equations (3.10a) and (3.10b) simply read

\[ |\Omega_+\rangle = \frac{1}{\sqrt{2}} (|g, n\rangle + |e, n - 1\rangle), \] (3.12a)

\[ |\Omega_-\rangle = \frac{1}{\sqrt{2}} (|g, n\rangle - |e, n - 1\rangle). \] (3.12b)

One of the advantages of the dressed-state basis is that time-evolution equations written in this basis are particularly simple. Indeed, the time-dependent states can be simply written as

\[ |\Omega_\pm(t)\rangle = e^{-i\Omega_\pm t}|\Omega_\pm\rangle(0), \] (3.13)
which can be easily used to determine the time-dependence of the bare (uncoupled) states

\[ |e, n - 1\rangle(t) = \cos \theta |\Omega_+\rangle(t) - \sin \theta |\Omega_-\rangle(t), \]

\[ |g, n\rangle(t) = \sin \theta |\Omega_+\rangle(t) + \cos \theta |\Omega_-\rangle(t). \]

Combining these equations with (3.13), (3.10a), (3.10b), (3.11) and performing some algebra we get

\[ |e, n - 1\rangle(t) = e^{-i(n-\frac{1}{2})\omega t} \times \left\{ \left( \cos \frac{1}{2}\omega_R t - i \sin \frac{1}{2}\omega_R t \cos 2\theta \right) |e, n - 1\rangle - i \sin 2\theta \sin \frac{1}{2}\omega_R t |g, n\rangle \right\}, \]

\[ |g, n\rangle(t) = e^{-i(n-\frac{1}{2})\omega t} \times \left\{ -i \sin 2\theta \sin \frac{1}{2}\omega_R t |e, n - 1\rangle + \left( \cos \frac{1}{2}\omega_R t + i \sin \frac{1}{2}\omega_R t \cos 2\theta \right) |g, n\rangle \right\}, \]

where \(|e, n - 1\rangle\) and \(|g, n\rangle\) denote the states taken at \(t = 0\). One feature of the atom-field evolution, important from the viewpoint of quantum information processing, is that interaction produces entanglement between the two subsystems. It can be easily verified, for instance, that for \(\Delta = 0\) and \(\omega_R t = \pi/2\) the state of the total system is maximally entangled.

Assuming now, as an illustration, that the initial state of the system is \(|e, n - 1\rangle\), the probability \(p_e(t)\) of finding the atom in the upper state \(|e\rangle\) is

\[ p_e(t) = 1 - \sin^2 \frac{1}{2} \omega_R t \sin^2 2\theta. \]

In resonance, this equation takes a simpler form,

\[ |e, n - 1\rangle(t) = e^{-i(n-\frac{1}{2})\Omega n t} \left\{ \cos \Omega_n t |e, n - 1\rangle - i \sin \Omega_n t |g, n\rangle \right\}, \]

\[ |g, n\rangle(t) = e^{-i(n-\frac{1}{2})\Omega n t} \left\{ -i \sin \Omega_n t |e, n - 1\rangle + \cos \Omega_n t |g, n\rangle \right\}, \]

and probability \(p_e(t)\) becomes simply

\[ p_e(t) = 1 - \sin^2 \frac{1}{2} g \sqrt{n} t = \cos^2 \frac{1}{2} g \sqrt{n} t, \]

while atomic inversion \(w(t) = \frac{1}{2}(p_e(t) - p_g(t))\) is

\[ w(t) = \frac{1}{2} - \sin^2 \frac{1}{2} g \sqrt{n} t. \]
These equations clearly demonstrate that atom-cavity interaction results in a reversible energy exchange between the two subsystems at Rabi frequency $\omega_R$ that depends on the number of photons. The smallest Rabi frequency at which the exchange occurs is the vacuum frequency that corresponds to $n = 1$ (the excitation is on the atomic side). The atom, initially prepared in $|e\rangle$, say, emits a photon and makes a transition to the state $|g\rangle$. A photon is then transferred back to the atom. As a consequence, the atom gets excited to the upper state and the whole situation repeats again and again with no energy losses provided cavity lifetime is infinite (in perfect cavities, where interacting system is perfectly isolated from environment). In real-life cavities, however, the oscillations are damped, so that for $t \to \infty$ $p_e(t)$ should decrease to zero (at zero temperature). Let us emphasize that equations (3.19) and (3.20) hold only when the atom, initially excited, enters into a cavity filled with exactly $n$ photons. If the number of photons is not fixed but fluctuates, then $p_e(t)$ and $w(t)$ need to be averaged over a probability distribution of photon numbers at $t = 0$. This is the case for, among others, coherent and thermal fields.

Let us now turn to a more complex scenario, where the atom interacts with a single-mode coherent field $|\alpha\rangle$,

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$  \hspace{1cm} (3.21)

characterized by average number of photons $|\alpha|^2$, uncertainty of the photon number $|\alpha|$, and Poissonian photon number distribution,

$$P_\alpha(n) = e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!}. \hspace{1cm} (3.22)$$

Let us assume the resonant case $\Delta = 0$, and the following initial state of the atom-field system:

$$|\psi\rangle = |e, \alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |e, n\rangle.$$

\hspace{1cm} (3.23)

In that case the probability to find the atom in its excited state is given by

$$p_e(t) = \sum_{n=0}^{\infty} P_\alpha(n) \cos^2 \Omega_n t = \sum_{n=0}^{\infty} P_\alpha(n) \frac{1 + \cos 2\Omega_n t}{2}. \hspace{1cm} (3.24)$$

Here one deals with a distribution of Rabi frequencies, therefore collapses and subsequent revivals of oscillations are expected to appear [30, 31]. Clearly, collapses and subsequent revivals are not peculiar to coherent fields only. Replacing Poisson distribution function $P_\alpha(n)$ in (3.24) by, for instance, the thermal probability function

$$P_{th}(n) = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}}, \hspace{1cm} (3.25)$$

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we get a formula appropriate for situations where the atom interacts with thermal fields.

All of this is well known and experimentally confirmed. Collapses and revivals in presence of an initial thermal field were first observed by Rempe et al [31].

### 3.2 Reducible approach

In this section we deal with the issue of Jaynes-Cummings type interaction of a two-level atom and a quantum field treated as an ensemble of $N$ indefinite-frequency oscillators. The central issue is to determine the way in which the inversion operator $R_3$ evolves in time.

In analogy to the irreducible case, the Jaynes-Cummings model Hamiltonian can be written as

$$ H = \hbar \Omega = \hbar \left( \omega_0 R_3 + \omega N_\omega + g R_+ a_\omega + g R_- a_\omega^\dagger \right). $$(3.26)

For definiteness, let us note that the atomic operators appearing above are products $R_3 \otimes I$ and $R_\pm \otimes I$, but for reasons of brevity we simply write them as $R_3$ and $R_\pm$, respectively.

It is advantageous to split the Hamiltonian into two commuting and Hermitian parts

$$ \Omega = \Omega_1 + \Omega_2, $$ (3.27)

where

$$ \begin{align*}
\Omega_1 &= \omega R_3 + \omega N_\omega, \\
\Omega_2 &= \Delta R_3 + g R_+ a_\omega + g R_- a_\omega^\dagger,
\end{align*} $$ (3.28, 3.29)

with $\Delta = \omega_0 - \omega$ and $\Omega_1$ and $\Omega_2$ being constants of motion, $[\Omega_1, \Omega] = [\Omega_2, \Omega] = 0$. Employing relations

$$ \begin{align*}
[\Omega_1, R_3] &= 0, \\
[R_3, \Omega_2] &= g R_+ a_\omega - g R_- a_\omega^\dagger,
\end{align*} $$ (3.30a, 3.30b)

the evolution of $R_3$ can be described by

$$ R_3(t) = e^{i\Omega t} R_3 e^{-i\Omega t} = e^{i\Omega_2 t} R_3 e^{-i\Omega_2 t}. $$ (3.31)

One way to proceed further is to decompose $\Omega_2$ into a sum of operators acting on eigenspaces of $I_N$ with creation and annihilation operators satisfying ordinary commutation relations with multiple of identity at the right-hand side, as discussed in section 2.5, that is

$$ \begin{align*}
\Omega_2 &= \sum_{s=0}^N \Omega_2(s) \pi_\omega(s/N) \\
&= \sum_{s=0}^N \left( \Delta R_3 + g R_+ a_\omega(s) + g R_- a_\omega^\dagger(s) \right) \pi_\omega(s/N).
\end{align*} $$ (3.32)
Due to the properties of projectors $\Pi_\omega(s/N)$ discussed in section 2.3, the operator $e^{\pm i\Omega_2 t}$ appearing in (3.31) can be written as

$$
e^{\pm i\Omega_2 t} = e^{\pm i\Omega_2(0)\pi_\omega(0/N)t} e^{\pm i\Omega_2(1)\pi_\omega(1/N)t} \ldots e^{\pm i\Omega_2(N)\pi_\omega(N/N)t}$$

$$= \sum_{s=0}^{N} e^{\pm i\Omega_2(s)t} \pi_\omega(s/N).$$  \hspace{1cm} (3.33)

Recall that we have already defined operator $\Pi(n)$ (see equation (2.88)) that projects onto the subspace of states of an ensemble of $N$ oscillators with exactly $n$ photons (of any frequency). Now, let us define two more operators appropriate in the context of the Jaynes-Cummings model, namely

$$\Pi_\omega(s, n) = \Pi(n)\pi_\omega(s/N),$$  \hspace{1cm} (3.34)

and

$$\hat{\Pi}_\omega(s, n) = \begin{cases} |e\rangle\langle e| \otimes \Pi_\omega(s, n - 1) + |g\rangle\langle g| \otimes \Pi_\omega(s, n), & n \geq 1 \\ |g\rangle\langle g| \otimes \Pi_\omega(s, 0), & n = 0 \end{cases}.$$  \hspace{1cm} (3.35)

Obviously

$$\sum_{n=0}^{\infty} \hat{\Pi}_\omega(s, n) = \pi_\omega(s/N),$$  \hspace{1cm} (3.36)

and

$$\hat{\Pi}_\omega(s, n)\hat{\Pi}_\omega(s', n') = \delta_{s,s'}\delta_{n,n'} \hat{\Pi}_\omega(s, n).$$  \hspace{1cm} (3.37)

With these operators $\Omega_2$ can be further decomposed as $\Omega_2 = \sum_{s=0}^{N} \sum_{n=0}^{\infty} \Omega_2(s, n)\hat{\Pi}_\omega(s, n)$, with $\Omega(s, n) = \Omega(s)\hat{\Pi}_\omega(s, n)$, and consequently

$$e^{\pm i\Omega_2 t} = \sum_{s=0}^{N} \sum_{n=0}^{\infty} e^{\pm i\Omega_2(s,n)t} \hat{\Pi}_\omega(s, N).$$  \hspace{1cm} (3.38)

As a simple example let us now explicitly write $\Omega_2(s, n)$ for $n = 0, 1$ and 2. We have, respectively,

$$\Omega_2(s, 0) = -\frac{\Delta}{2} |g\rangle\langle g| \otimes \Pi(s, 0) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\Delta}{2} \Pi_\omega(s, 0) \end{pmatrix},$$

$$\Omega_2(s, 1) = \begin{pmatrix} \frac{\Delta}{2} \Pi_\omega(s, 0) & ga_\omega(s)\Pi_\omega(s, 1) \\ ga_\omega^\dagger(s)\Pi_\omega(s, 0) & -\frac{\Delta}{2} \Pi_\omega(s, 1) \end{pmatrix},$$

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\[ \Omega_2(s, 2) = \begin{pmatrix} \frac{\Delta^2}{4} \Pi_{\omega}(s, 1) & g a_{\omega}(s) \Pi_{\omega}(s, 2) \\ g a_{\omega}^+(s) \Pi_{\omega}(s, 1) & -\frac{\Delta^2}{4} \Pi_{\omega}(s, 2) \end{pmatrix}. \]

These solutions can be easily generalized as

\[ \Omega(s, n) = \begin{pmatrix} \frac{\Delta^2}{4} \Pi_{\omega}(s, n - 1) & g a_{\omega}(s) \Pi_{\omega}(s, n) \\ g a_{\omega}^+(s) \Pi_{\omega}(s, n - 1) & -\frac{\Delta^2}{4} \Pi_{\omega}(s, n) \end{pmatrix}, \]

and

\[ \Omega_2(s, n)^{2j} = \left( \sqrt{\frac{\Delta^2}{4} + g^2 n \frac{s}{N}} \right)^{2j} \Pi_{\omega}(s, n), \]

\[ \Omega_2(s, n)^{2j+1} = \frac{\left( \sqrt{\frac{\Delta^2}{4} + g^2 n \frac{s}{N}} \right)^{2j+1}}{\sqrt{\frac{\Delta^2}{4} + g^2 n \frac{s}{N}}} \Omega_2(s, n) \Pi_{\omega}(s, n). \]

Making use of these relations, equation (3.38) becomes

\[ e^{\pm i \Omega_2 t} = \sum_{s=0}^{N} \sum_{n=0}^{\infty} \cos \sqrt{\frac{\Delta^2}{4} + g^2 n \frac{s}{N}} t \Pi_{\omega}(s, n) \]

\[ \pm \sum_{s=0}^{N} \sum_{n=0}^{\infty} i \sin \sqrt{\frac{\Delta^2}{4} + g^2 n \frac{s}{N}} t \Omega_2(s, n) \Pi_{\omega}(s, n). \] (3.40)

Defining the operator

\[ X = \sum_{s=0}^{N} \sum_{n=0}^{\infty} n \frac{s}{N} \Pi_{\omega}(s, n) = R_3 L_\omega + \frac{1}{2} L_\omega + a_\omega^+ a_\omega, \] (3.41)

one arrives at a more concise form, namely

\[ e^{\pm i \Omega_2 t} = \cos \sqrt{\frac{\Delta^2}{4} + g^2 X} t \pm i \frac{\sin \sqrt{\frac{\Delta^2}{4} + g^2 X} t}{\sqrt{\frac{\Delta^2}{4} + g^2 X}} \Omega_2. \] (3.42)

Inserting the last equation into (3.31) and employing relations

\[ [X, R_3] = [X, \Omega_1] = [X, \Omega_2] = 0, \] (3.43a)

\[ [X, R_\pm] = \pm R_\pm L_\omega, \] (3.43b)

\[ [X, R_- a_\omega^+] = [X, R_+ a_\omega] = 0, \] (3.43c)
we finally get

\[ R_3(t) = R_3 \left( 1 - 2g^2X \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + g^2X} t}{\frac{\Delta^2}{4} + g^2X} \right) \]

\[ + \left( \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + g^2X} t \Delta}{\frac{\Delta^2}{4} + g^2X} - i \frac{\sin 2 \sqrt{\frac{\Delta^2}{4} + g^2X} t}{2 \sqrt{\frac{\Delta^2}{4} + g^2X}} \right) gR_+ a_\omega. \]

\[ + \left( \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + g^2X} t \Delta}{\frac{\Delta^2}{4} + g^2X} + i \frac{\sin 2 \sqrt{\frac{\Delta^2}{4} + g^2X} t}{2 \sqrt{\frac{\Delta^2}{4} + g^2X}} \right) gR_- a_\omega^\dagger. \] (3.44)

Given this equation we are now interested in examining the way the atomic inversion evolves in time for various initial atom-field states. First, let us take the field initially prepared in the vacuum state \(|\Omega\rangle\). Taking an average of (3.44) in \(|\Omega\rangle\) we get

\[ R_{\text{vac}}(t) = \langle \Omega | R_3(t) | \Omega \rangle = R_3 - g^2 |e\rangle \langle e| \sum_{s=0}^{N} \frac{s}{N} \sin^2 \frac{\sqrt{\frac{\Delta^2}{4} + g^2 \frac{s}{N}} t}{\frac{\Delta^2}{4} + g^2 \frac{s}{N}} \langle \Omega | \Omega_{\omega}(s/N) | \Omega \rangle, \] (3.45)

which allows to calculate the atomic evolution for arbitrary initial atomic states. Just as we might expect, the evolution is trivial for \(|g, \Omega\rangle\), as \(\langle g, \Omega | R_3(t) | g, \Omega \rangle = -\frac{1}{2}\), and is independent of time. Taking, on the other hand, the initial state of the system to be \(|e, \Omega\rangle\), equation (3.45) leads to

\[ w(t) = \langle e, \Omega | R_3(t) | e, \Omega \rangle = \frac{1}{2} - \sum_{s=0}^{N} g^2 \frac{s}{N} \sin^2 \frac{\sqrt{\frac{\Delta^2}{4} + g^2 \frac{s}{N}} t}{\frac{\Delta^2}{4} + g^2 \frac{s}{N}} \binom{N}{s} Z_{\omega}(1 - Z_{\omega})^{N-s}, \] (3.46)

where

\[ \omega_R(s) = \sqrt{\frac{\Delta^2}{4} + g^2 \frac{s}{N}}, \] (3.47)

can be easily identified as an elementary Rabi frequency. Equation (3.46) describes the reducible-representation form of vacuum Rabi oscillations. Before discussing details of this solution let us look at an analogous formula valid in various irreducible representations. As an example, let us consider representation \([a_\omega, a_\omega^\dagger] = \delta_{\omega,\omega'} Z, Z \in \mathbb{R}, \) with some carrier Hilbert space containing the vacuum state \(|0\rangle\). Omitting detailed calculations, it can be shown that \(R_3(t)\) is
\[ R_3(t) = R_3 \left( 1 - 2 \tilde{g}^2 \tilde{X} \frac{\sin^2 \left( \frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X} \right)}{\frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X}} \right) \]
+ \left( \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X}}}{\frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X}} \frac{\Delta}{2} \frac{i}{2} \right) \tilde{g} R_+ \tilde{a}_\omega 
+ \left( \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X}}}{\frac{\Delta^2}{4} + \tilde{g}^2 \tilde{X}} \frac{\Delta}{2} \frac{i}{2} \right) \tilde{g} R_- \tilde{a}_\omega, \tag{3.48} \]

with \( \tilde{a}_\omega = a_\omega / \sqrt{Z} \), \( \tilde{a}_\omega^\dagger = a_\omega^\dagger / \sqrt{Z} \), \( \tilde{g} = \sqrt{Z} g \), and \( \tilde{X} = R_3 + \tilde{a}_\omega^\dagger \tilde{a}_\omega + \frac{1}{2} \). For the atom-field system initially in \(|e, 0\rangle\) the average of the inversion operator is

\[ w(t) = \frac{1}{2} - \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + \tilde{g}^2}}{\frac{\Delta^2}{4} + \tilde{g}^2} \frac{\Delta}{2} \frac{t}{4} \]. \tag{3.49} \]

As we can see, the only difference between various irreducible representations lies in the values of the coupling constant \( \tilde{g} = \sqrt{Z} g \). Since \( g \) is proportional to the electron charge, different irreducible representations effectively differ by the value of the electron charge \( e_0 / \sqrt{Z} \). This type of rescaling is exactly what occurs in transition from the bare charge \( e_0 \) to the physical, renormalized charge \( e_{\text{ph}} = e_0 \sqrt{Z} \), typical of renormalized electromagnetic fields. Therefore, all irreducible representations produce the same predictions up to the value of \( Z \).

What makes (3.46) different from (3.49) is, in turn, the fact that the former consists of \( N \) different elementary frequencies, so that the dynamics must involve collapses and revivals. Although it contradicts intuitions based on irreducible representations (superpositions of oscillations are characteristic of nontrivial photon-number distributions), we will see in the following chapters that it does not contradict experimental results.

It is important to realize that superpositions of \( N \) different oscillations occur only for \( N < \infty \). In the extreme case of infinite number of oscillators, \( N \to \infty \), the law of large numbers applied to the binomial distribution (at the right-hand side of (3.46)) implies that the fraction \( s/N \) of oscillators in mode \( \omega \) concentrates around \( Z_\omega \), and the atomic inversion (3.46) becomes

\[ \lim_{N \to \infty} w(t) = \frac{1}{2} - \frac{g^2 Z_\omega}{\left( \frac{\Delta^2}{4} + g^2 Z_\omega \right)} \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + g^2 Z_\omega} \frac{t}{2}}{4} \]. \tag{3.50} \]

a form characteristic of irreducible representations. Formulas (3.49) and (3.50), corresponding to atomic inversion in irreducible and reducible formalisms, respectively, appear so similar that a substantial difference can be easily overlooked. Namely, the parameter \( Z \) in (3.49)
Figure 3.1: Probability to find the atom in state $|g\rangle$ predicted by the reducible representation theory. An initially excited atom interacts with a vacuum field $|Q\rangle$ in an ideal cavity (no dissipation is assumed) for $N = 10^3$ (plot A) and $N = 10^4$ (plot B). Other parameters are: $\Delta = 0$, $g_{\text{ph}} = 1$ kHz, $Z = 0.1$. In the time interval used in these figures (unaccessible in contemporary experiments) Rabi oscillation consists of successive collapses and revivals, in contrast to the oscillation at a single frequency predicted by the standard theory. Vacuum Rabi oscillations at smaller time scales (tens of milliseconds) are depicted in figure 3.2.

is a constant independent of $\omega$, whereas $Z_{\omega}$ in (3.50) is a value of a probability function which by definition vanishes for $\omega \to \infty$. Defining $Z = \max_{\omega\in\mathbb{R}}\{Z_{\omega}\}$ and $Z_{\omega} = Z \chi_{\omega}$, we note that formula (3.50) involves renormalized (physical) coupling $g_{\text{ph}} = g\sqrt{Z}$ and the cutoff function $\chi_{\omega} = Z_{\omega}/Z$, $0 \leq \chi_{\omega} \leq 1$ regularizing it for extremely high frequencies $\omega$. It is important to realize that we did not introduce any cutoff in the Hamiltonian (a standard trick to deal with infinities within the irreducible approach). The cutoff appears here automatically and follows from the structure of the vacuum state typical of the reducible representation. In fact, the theory gets regularized even though it is not needed in such a simple example. The reducible-representation asymptotic formula reconstructs exactly the standard one if we assume that $\chi_{\omega} = 1$ for frequencies belonging to the optical regime. The limit $N \to \infty$ thus appears to play a role of a correspondence principle allowing to pass from one formulation to the other.

Notice that for $N$ large enough, small $Z$ and $\chi_{\omega} = 1$, the binomial distribution in (3.46) can be approximated as

$$
\left(\begin{array}{c}
N \\
s
\end{array}\right) Z_s^s (1 - Z_{\omega})^{N-s} \approx \left(\begin{array}{c}
N \\
s
\end{array}\right) Z_s^s (1 - Z)^{N-s} \approx \frac{e^{(s-NZ)^2/2NZ}}{\sqrt{2\pi NZ(1-Z)}} \approx \frac{e^{(s-NZ)^2/2NZ}}{\sqrt{2\pi NZ}}.
$$

In addition, $g^2/N$ can be written as $g_{\text{ph}}^2 s/(NZ)$, so that the parameter that effectively controls Rabi oscillations is the product $NZ$. Figures 3.1 and 3.2 show vacuum Rabi oscillations for various $N < \infty$ and $Z = 0.1$, $Z = 0.01$, respectively.
Now, let us proceed to the field prepared initially in a monochromatic coherent state \( |\alpha\rangle = D(\alpha)|\Omega\). In this case the reduced atomic inversion becomes

\[
R_{\text{coh}}(t) = \langle \alpha | R_3(t) | \alpha \rangle
= R_3 + \sum_{s=0}^{N} \frac{s}{N} \sum_{n=0}^{\infty} \langle \alpha | \Pi_\omega(s, n) | \alpha \rangle \times \left\{ \begin{array}{c}
g^2 n |g \rangle \langle g| \frac{\sin^2 \frac{\Delta^2}{4} + g^2 n \frac{s}{N}}{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}} - g^2 (n+1) |e\rangle \langle e| \frac{\sin^2 \frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}} \\
+ g \alpha R_+ \left( \frac{\sin^2 \frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}} \frac{t \Delta}{2} - i \frac{\sin 2 \frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}{2 \sqrt{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}} t \right) \\
+ g \alpha^* R_- \left( \frac{\sin^2 \frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}} \frac{t \Delta}{2} + i \frac{\sin 2 \frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}{2 \sqrt{\frac{\Delta^2}{4} + g^2 (n+1) \frac{s}{N}}} t \right) \end{array} \right. \right\}.
\]

The average \( \langle \alpha | \Pi_\omega(s, n) | \alpha \rangle \), appearing in the second row, can be evaluated as follows

\[
\langle Q | D^\dagger(\alpha) \Pi_\omega(s, n) D(\alpha) | Q \rangle = \langle Q | e^{-|\alpha|^2 L_\omega} e^{\omega \pi \Pi_\omega(s, n) e^{\omega \pi} | Q \rangle \\
= \frac{|\alpha|^{2n}}{n!^2} \langle Q | e^{-|\alpha|^2 L_\omega} a^\dagger_\omega^{n} \Pi_\omega(s, n) a^\dagger_\omega^{n} | Q \rangle \\
= \frac{|\alpha|^{2n}}{n!^2} \langle Q | e^{-|\alpha|^2 L_\omega} a^\dagger_\omega^{n} \sum_{m=0}^{\infty} \Pi_\omega(s, m) a^\dagger_\omega^{m} | Q \rangle \\
= \frac{|\alpha|^{2n}}{n!^2} \langle Q | e^{-|\alpha|^2 L_\omega} a_\omega(s)^n \pi_\omega(s/N) a^\dagger_\omega(s)^n | Q \rangle \\
= \frac{|\alpha|^{2n}}{n!^2} e^{-|\alpha|^2 \frac{s}{N} \pi \pi_\omega(s/N) a_\omega(s)^n a^\dagger_\omega(s)^n | Q \rangle.
\]

Making use of the relation

\[
[a_\omega(s), a^\dagger_\omega(s)]^n = n \frac{s}{N} a^\dagger_\omega(s)^{n-1},
\]

equation (3.53) can be recast as

\[
\langle Q | D^\dagger(\alpha) \Pi_\omega(s, n) D(\alpha) | Q \rangle = n! \left( \frac{s}{N} \right)^n \frac{|\alpha|^{2n}}{n!^2} e^{-|\alpha|^2 \frac{s}{N} \pi \pi_\omega(s/N) | Q \rangle \\
= \frac{1}{n!} \left( \frac{|\alpha|^{2s}}{N} \frac{s}{N} \right)^n e^{-|\alpha|^2 \frac{s}{N}} \left( \frac{N}{s} \right) Z^s_\omega (1 - Z_\omega)^{N-s}.
\]

(3.55)
Suppose now that the coherent state interacts with an excited atom. The reducible representation approach leads then to a somewhat complicated equation

\[ w(t) = \langle e, \alpha | R_3(t) | e, \alpha \rangle = \frac{1}{2} - \sum_{n=0}^{\infty} g^2(n + 1) \frac{\Delta^2}{4 + g^2(n + 1) s N} t \times \left( \frac{|\alpha|^2}{n!} \right)^n \left( \frac{N}{s} \right)^n \frac{\sin^2 \sqrt{\Delta^2 + g^2(n + 1) s N} t}{\Delta^2 + g^2(n + 1) s N} \left( e^{-|\alpha|^2} \right) \left( \frac{1}{s} \right) Z^s_\omega (1 - Z_\omega)^{N-s}, \]  

which, however, becomes familiar in the already mentioned limit \( N \to \infty \). Indeed, we get

\[ \lim_{N \to \infty} w(t) = \frac{1}{2} - \sum_{n=0}^{\infty} g^2_{\text{ph}} \chi_\omega (n + 1) \frac{\Delta^2 + g^2_{\text{ph}} \chi_\omega (n + 1) t}{\Delta^2 + g^2_{\text{ph}} \chi_\omega (n + 1)} \left( |\alpha_{\text{ph}}|^2 \chi_\omega \right)^n \frac{1}{n!} e^{-|\alpha_{\text{ph}}|^2 \chi_\omega}, \]  

where we have used the renormalized coupling \( g_{\text{ph}} = g \sqrt{Z} \) and introduced the renormalized amplitude \( \alpha_{\text{ph}} = \alpha \sqrt{Z} \). For reasons similar to those discussed previously, equation (3.57) coincides with the standard formula but with cutoff function \( \chi_\omega \) and renormalized coupling and field amplitude. Obviously, this modification is negligible if \( \chi_\omega \approx 1 \), which we assume in the region of interest. This result supports the observation made below equation (3.50): The reducible representation approach recovers the standard one in the limit \( N \to \infty \). All these facts are well visible when one depicts the dependence on \( N \) of probability of finding the atom in \( |g\rangle \), as shown in figure 3.3. The idea behind these figures is clear: For a given interval of time, one can achieve as small discrepancy between the two formulations as one wants. The key factor is the value of \( N \).

### 3.3 Hilbert space partitioning

Recall that in accordance with equation (2.68), any state of the field system can be decomposed into a sum of orthogonal states belonging to subspaces \( \mathcal{H}(s) = \pi_{s/N} \mathcal{H} \), \( s = 0, \ldots, N \). The observation that these spaces are invariant under the action of the Jaynes-Cummings Hamiltonian (3.26) provides an exceptionally simple method of dealing with the reducible representation calculations. Informally, the procedure can go as follows:

1. Describe the dynamics of the system in \( \mathcal{H}(s) \) keeping \( s \) arbitrary but fixed. As discussed earlier, each subspace \( \mathcal{H}(s) \) corresponds to an irreducible representation of the CCR algebra and thus solutions known from standard treatment can be exploited.

2. As the exact number of oscillators in a given mode in question is uncertain, the global dynamics decomposes into dynamics in all subspaces \( \mathcal{H}(s) \), \( s = 0, 1, \ldots, N \). The probability to enter a specific subspace is described by the binomial distribution \( b(s; Z_\omega, N) \).
3. The overall dynamics is an average over consecutive subspaces $\mathcal{H}(s)$, $s = 0, 1, 2, \ldots, N$.

As an illustration, let us apply this procedure to the Jaynes-Cummings model. It is obvious that the Hamiltonian (3.26) restricted to $\mathcal{H}(s)$ is $H(s) = \hbar \Omega(s)$, with

$$\Omega(s) = \omega_0 R_3 + \omega N_\omega \pi_\omega(s/N) + g_\omega R_+ a_\omega(s) + g_\omega R_- a_\omega^\dagger(s).$$

Given this formula, we may formally repeat all the steps known from the irreducible treatment of the model. Assuming the initial state of the system is $\pi_\omega(s/N)|e, O\rangle$, the atomic inversion is simply (3.49) with $Z$ replaced by $s/N$, that is

$$w(t)\big|_{\text{fixed } s} = \frac{1}{2} - g^2 \frac{s}{N} \frac{\sin^2 \sqrt{\Delta^2 + g^2 s/N} t}{\Delta^2 + g^2 s/N}. \tag{3.58}$$

In the extreme case $s = 0$ the right-hand side of this equation equals $1/2$ which means that the atom is trapped in its initial state $|e\rangle$.

Solution (3.58) is valid for a fixed $s$. The overall atomic inversion is the sum of partial solutions valid in subspaces $\mathcal{H}(s)$, multiplied by $b(s; Z_\omega, N)$. Putting all these facts together we successfully recover formula (3.46). Analogous reasoning may be repeated for any initial atom-field state.

The knowledge of this gives us the ability to look at the correspondence principle from a fresh viewpoint. In the asymptotic case, $N \to \infty$, the fraction $s/N$ of oscillators existing in the frequency state $|\omega\rangle$ approaches $Z_\omega$. It means that there is only one possibility for the number of oscillators in mode $\omega$, and it is $s = Z_\omega N$ (the probability to get any other value of $s$ equals 0). In that case, the dynamics of the system happens in $\mathcal{H}(Z_\omega N)$ and, as expected, equation (3.46) transforms into (3.50). It looks as if $N \to \infty$ somehow reduced the number of possible field configurations from $N + 1$ ($s = 0, 1, 2, \ldots, N$) to one ($s = Z_\omega N$).
Figure 3.2: Correspondence principle for vacuum Rabi oscillations in an ideal cavity. Probability to find the atom in the lower state $|g\rangle$ predicted by the standard theory (dashed) and the reducible one (solid). Parameters are $\Delta = 0$, $g_{\text{ph}} = 1\text{kHz}$, $Z = 0.01$. $N$ equals respectively (A) $10^2$, (B) $10^3$, (C) $10^4$, (D) $10^5$, (E) $10^6$. The tendency is well apparent: The greater value of $N$, the smaller deviations between predictions of both formalisms in a given time interval. In the limit $N \to \infty$ the atomic inversion agrees with the one obtained in the standard treatment in any time interval. Comparison with figure 3.1 allows us to state that analysis of these phenomena at different time scales may be of great importance within an experimental context.
Figure 3.3: Correspondence principle for monochromatic coherent field. Time evolution of the atomic inversion of an initially excited atom interacting with a coherent field in an ideal cavity. The dashed curve represents exact theoretical predictions of the standard theory, while the solid curve predictions of the reducible one. In all five cases the parameters are: $\Delta = 0$, $g_{ph} = 1$ kHz, $Z_\omega = 0.1$ and $\alpha_{ph} = 3$ (the average number of photons in the cavity is equal to $|\alpha_{ph}|^2 = 9$). The number $N$ of field oscillators varies between $10^2$ and $10^6$, and equals (A) $10^2$, (B) $10^3$, (C) $10^4$, (D) $10^5$, (E) $10^6$, respectively. For $N = 10^5$ the agreement between these two solutions is remarkable, while for $N = 10^6$ the solutions are almost indistinguishable. It is clearly visible that for fixed interval of time we may achieve as high accuracy as we want by increasing $N$, still keeping $N < \infty$. 
Chapter 4
Jaynes-Cummings model in lossy cavities

In nearly every detective novel since the admirable stories of Conan Doyle there comes a time when the investigator has collected all the facts he needs for at least some phase of his problem. These facts often seem quite strange, incoherent, and wholly unrelated. The great detective, however, realizes that no further investigation is needed at the moment, and that only pure thinking will lead to a correlation of the facts collected. So he plays his violin, or lounges in his armchair enjoying a pipe, when suddenly, by Jove, he has it! Not only does he have an explanation for the clues at hand, but he knows that certain other events must have happened. Since he now knows exactly where to look for it, he may go out, if he likes, to collect further confirmation for his theory.

A. Einstein and L. Infeld, The evolution of physics

Dynamics of atom-field systems is, in real-life setups, affected by non-ideality of cavities. Finite quality $Q$-factors, which are proportional to cavity field lifetimes $T_{\text{cav}}$, cause leakage of photons from cavities or, alternatively, decrease numbers of photons without corresponding increases in atomic populations, as it would be the case in ideal cavities. In order to get a proper picture of phenomena we need to know how to compare theory and experimental data.

4.1 Atom-field interaction experiments

The idea of atom-field experiments in strong-coupling regime is to inject atoms prepared in a given state into a suitably prepared cavity, and letting them to interact for a given amount of time [32, 33, 34, 35]. The interaction time between the atom and the cavity is adjusted by selecting appropriate velocities of atoms. Measurable quantities, such as atomic inversion at a given $t$, are inferred from a collection of results obtained for large numbers of atoms moving at identical velocities, and examined at the exit of the cavity.
In 1996 the group from Laboratoire Kastler–Brossel in Paris reported to date most precise measurements of vacuum Rabi oscillations in cavity QED [11]. The suggestive plots from [11] have achieved status of classic textbook illustrations of agreement between theory and experiment in quantum optics [28, 36]. The experimental setup is close to ideal, and as such, is a promising candidate for implementation of quantum logic gates. This brings, however, all the issues one will encounter in a realistic quantum computer, with qubit-reservoir interactions and various kinds of decoherence in the first place [37, 38, 39, 40].

The cavity used in this particular experiment is an open Fabry-Perot resonator consisting of two mirrors of diameter $d = 50$ mm each, and with mirror spacing $L = 27$ mm measured on the cavity axis. The mode sustained by the cavity has a Gaussian profile characterized by the waist $w = 5.96$ mm and centered in the middle of the cavity with maximal value $g = 47\pi 10^3$ Hz. The walls of the cavity (mirrors) are cooled to approximately $0.8$ K which corresponds to the average number $\bar{n} = 0.05$ of photons of resonant frequency. The two-level system that interacts with the cavity field is implemented by circular Rydberg states of rubidium, with principal quantum numbers $n = 50$ and $n = 51$, playing the roles of, respectively, lower, $|g\rangle$, and upper, $|e\rangle$, levels. The transition frequency between these states is $\omega_0 \approx 2\pi 51.099$ GHz and the inverse radiative lifetime $1/\tau \approx 0.03$ kHz. The cavity relaxation time, on the other hand, is $T_{\text{cav}} = 220\mu$s. The parameters of this experiment fulfill conditions of the strong-coupling regime (reported cavity $Q$-parameter is of the order of $7 \times 10^7$).

The experimental setup is schematically shown in figure 4.1. Assuming the time of flight between the source ($S$) and the detector ($D$) is $t$, the atom arrives at the center of the cavity at $t/2$. Relation between the distance $d$ and the time $t$ is given by $d = vt$, where $v$ is the velocity of the atom. The run of the experiment that measures the atomic state at time $t$ corresponds to the Hamiltonian $H(t')$, $0 \leq t' \leq t$, whose coupling is given by $g(t') = ge^{-v^2(t/2-t')^2/w^2}$. Let us note that because of the non-trivial structure, the atom-field coupling during the interaction is almost all the time smaller than the maximal value $g = g(t/2)$. One expects therefore that the Rabi oscillation (at exact resonance, $\Delta = 0$) is not performed with the Rabi frequency $g$ but with some smaller effective frequency $g_{\text{eff}}$. In fact, Brune et al. assume that this frequency should be computed according to the formula

$$g_{\text{eff}}t = \int_0^t g(t')dt' \approx g \int_{-\infty}^{\infty} e^{-(vt')^2/w^2} dt' = g\sqrt{\pi} \frac{w}{v} = g\sqrt{\pi} \frac{w}{d} t = g t_{\text{eff}}, \quad (4.1)$$
Figure 4.2: Probability of the atomic ground state as a function of the effective time $t_{\text{eff}}$. Dots with error bars represent experimental points. (A) The dashed line represents (4.2) with $\gamma^{-1} = 220 \cdot 10^{-6}$s; the solid line corresponds to the corrected formula (4.4), hence after unifying the notion of time. (B) We have simply multiplied the probabilities known from the ideal Jaynes-Cummings model by exponential factors $\exp(-\gamma t_{\text{eff}})$ (dashed line) and $\exp\left(-\frac{\gamma}{\sqrt{\pi} w/d} t_{\text{eff}}\right)$ (solid line). In both cases $g = 47\pi 10^3$Hz and $T = 0.8$K. Thermal distribution is expanded up to $n = 2$.

which shows that the effective coupling is $g_{\text{eff}} = g\sqrt{\pi} w/d$ or $g_{\text{eff}} \approx 0.2g$ for parameters characteristic of the experiment in question. The second observation is that effective frequency is dual to the effective time, $t_{\text{eff}}$. Both the book [32] and the review [41] make it very clear that the data reported in [11] were plotted as functions of the effective time $t_{\text{eff}}$, and not as those of the real evolution time $t$.

The data obtained in the experiment is presented in figure 4.2. Dots with error bars represent experimental points corresponding to the probability of finding the atom in the lower state $|g\rangle$. The solid and dashed curves represent fits by means of damped sinusoids [11], i.e. the function

$$p_g^{\text{exp}}(t_{\text{eff}}) = 1 - \frac{1}{2} \sum_{n=0}^{\infty} P(n)\left(1 + e^{-\gamma t_{\text{eff}}} \cos(2g\sqrt{n + 1} t_{\text{eff}})\right),$$

(4.2)

for two different damping parameters $\gamma$ and with $P(n)$ being the thermal distribution function. The maximum value of $t_{\text{eff}}$ is 90$\mu$s and corresponds to approximately 426$\mu$s of true interaction time.

Fitting formula (4.2) has caused a lot of confusion in the literature, and there were two main reasons for that. One reason was that the first paper [11] did not clearly state that the time axis involved $t_{\text{eff}}$ and not $t$. The second reason can be understood after having solved master equations with time dependent Hamiltonians and non-constant coupling $g(t')$. We will see that inclusion of the Gaussian structure of $g(t')$ effectively changes the term $gt$, typical of the constant coupling, into $g_{\text{eff}} t = g t_{\text{eff}}$, but does not affect the terms $\gamma t$. It follows that if we plot the experimental data in the true time $t$, the fitting function should read
\[ p_g^{\text{exp}}(t) = 1 - \frac{1}{2} \sum_{n=0}^{\infty} P(n) \left( 1 + e^{-\gamma t} \cos(2g_{\text{eff}} \sqrt{n+1} t) \right). \]  

(4.3)

But if we decide to rescale the time from \( t \) to \( t_{\text{eff}} = t \sqrt{\pi w/d} \), as it was done in [11], then the fitting formula becomes

\[ p_g^{\text{exp}}(t_{\text{eff}}) = 1 - \frac{1}{2} \sum_{n=0}^{\infty} P(n) \left( 1 + e^{-\gamma/t_{\text{eff}}} \cos(2g_{\text{eff}} \sqrt{n+1} t_{\text{eff}}) \right), \]  

(4.4)

which was, however, not clearly stated in [11]. In other words, the effective damping parameter is \( \gamma/(\sqrt{\pi w/d}) \) and not \( \gamma \). Many authors, including ourselves [5, 42, 43, 44], have noticed that the data presented in [11] seem to correspond to a cavity whose lifetime is 40-50\,\mu s, and not 220\,\mu s as claimed in [11]. It looked like the cavity was five times worse than it was assumed. However, when we set the cavity damping parameters to be \( \gamma/(\sqrt{\pi w/d}) \) we find that the effective lifetime of the cavity is around 46\,\mu s, in agreement with the data.

In figure 4.2A the time axis corresponds to \( t_{\text{eff}} \). The dashed line is the fit with \( \gamma t_{\text{eff}} \) where \( 1/\gamma = 220 \,\mu s \); the solid curve is damped by \( \gamma/(\sqrt{\pi w/d}) t_{\text{eff}} \), with the same value of \( \gamma \). As we can see, an explanation of this discrepancy may be trivial.

It is important to know, however, that, in spite of the simple explanation of the discrepancy, the fitting formulas cannot be strictly regarded as representing theoretical predictions, and it is easy to understand why. The long-time asymptotics of both (4.3) and (4.4) is

\[ \lim_{t_{\text{eff}} \to \infty} p_g^{\text{exp}}(t_{\text{eff}}) = \lim_{t \to \infty} p_g^{\text{exp}}(t) = \frac{1}{2}, \]  

(4.5)

whereas it is obvious that after taking dissipation into account this probability should asymptotically reach unity.

Notice that having solutions of the Jaynes-Cummings model with no dissipation, and wanting to explicitly incorporate it, we could simply multiply probability \( p_e(t) \) by the exponential \( \exp(-\gamma t) \) that simulates dissipation in the system. This would be the simplest, we may say, operational model of dissipation in the Jaynes-Cummings framework with the reasonable property that in the limiting case, \( t \to \infty \), the probability \( p_g(t) = 1 - e^{-\gamma t} p_e(t) \) would asymptotically approach unity. This is not, however, the case for (4.2). To understand this feature, recall that the atomic inversion is defined as \( w(t) = \frac{1}{2} (p_e(t) - p_g(t)) \). Upon substituting \( p_e(t) \) and \( p_g(t) \) obtained for the ideal Jaynes-Cummings model at \( T = 0 \,\text{K} \), and for \( |e, 0\rangle \) being the initial state of the system, we get \( w(t) = \frac{1}{2} (p_e(t) - p_g(t)) = \frac{1}{2} \cos(2g t) \). Multiplying the right-hand side of this expression by \( \exp(-\gamma t) \), and solving with respect to \( p_g(t) \), we get equation (4.3) taken for the vacuum field. The analogous calculations can be carried out for any initial photon number distribution. This means that the damped sinusoids approach is based on the assumption that it is the inversion which vanishes for long times, instead of probability of finding the atom in \( |e\rangle \). It would be an error to try to predict a long-time atom-field behavior on the basis of this fitting scheme.

We will later see that recipe (4.1) works indeed for system-reservoir interactions involving dressed-state Davies operators, but in fact is not exactly true for the so called phenomenological model.
4.2 Dissipative and nondissipative decoherence

There are two main sources of decoherence that were identified in the literature in the context of the experiment of Brune et al. [11].

The analysis of dissipation based on quantum trajectories approach [43, 44] leads to the conclusion that damping due to energy loss in the cavity should have the form $p_{\kappa,e}(t) = e^{-\kappa t}p_e(t)$, where $p_e(t)$ is the probability of finding the atom in the excited state in an ideal cavity, and $2\kappa = 1/T_{\text{cav}}$. The factor 2 takes into account the fact that energy is not dissipated if the atom is in the excited state and there is no photon in the cavity. Obviously, for $t \to \infty$ the atom is with certainty found in its ground state.

Bonifacio et al. [42, 45] identified a nondissipative decoherence, that is a kind of decoherence that does not result from unpredictable interaction of a system at hand with environment, but from fluctuations of classical parameters or internal variables of the system. The point where this type of decoherence appears is in averaging over individual experimental runs, where each of them corresponds to random values of some parameters. Under certain conditions, the nondissipative decoherence may be present even in isolated systems, where the undesirable interaction with environment is absent. In the context of cavity QED, it may result from the lack of knowledge about the precise time of interaction between atoms that fly through the cavity and the cavity itself. This uncertainty is due to randomness of the detection time which, in turn, follows from uncertainty of atomic velocities. This means that the data collected at time $t$ should not be compared directly with $\rho(t)$ describing the state computed on the basis of first principles, but with the average

$$\rho_{\Delta t}(t) = \int_0^\infty dt' p_{\Delta t}(t,t')\rho(t'),$$

where $\rho(t')$ is the first-principles state, and $p_{\Delta t}(t,t')$ describes our lack of knowledge as to the exact duration of time evolution. The data from [11] involve a sample of 90 points selected from the time interval $0 < t_{\text{eff}} < 90\,\mu s$. Therefore the time-of-measurement uncertainty may be assumed to satisfy $0 < \Delta t < 1\,\mu s$, which indeed turns out to reasonably model the data. The probability distribution introduced in [42] is

$$p_{\Delta t}(t,t') = \frac{e^{-t'/\Delta t}(t'/\Delta t)^{t'/\Delta t-1}}{\Gamma(t'/\Delta t)},$$

with the natural property $\lim_{\Delta t \to 0} p_{\Delta t}(t,t') = \delta(t-t')$ implying $\rho_{\Delta t}(t)|_{\Delta t=0} = \rho(t)$.

In all the representations discussed in this thesis we have arrived at atomic probabilities involving terms of the form

$$p_e(t) = A + B \sin^2 \omega_R t.$$

Irreversible leakage of photons out of a cavity can be modeled, in the simplest case, by multiplying $p_e(t)$ by exponential factor $e^{-\kappa t}$. Taking this into account, and making use of linearity of equation (4.6), we get
Figure 4.3: Predictions of the standard theory based on irreducible representations of CCR for an atom initially in a mixed state with $p_+ = 0.99$ and a thermal field with $\bar{n} = 0.05$ (corresponds to $T = 0.8\,K$ inside of the cavity). Plots in (A) correspond to equation (4.9) multiplied by exponential factor $e^{-\kappa t}$ with $\kappa = 1/440 \,10^6\,Hz$ (solid) and $\kappa = 1/80 \,10^6\,Hz$ (dashed). Plots (B) in addition take into account randomness of the atom-field interaction time with $\Delta t = 0.5\,\mu s$, while damping factors are $\kappa = 1/80 \,10^6\,Hz$ (dotted), $\kappa = 1/440 \,10^6\,Hz$ (dashed) and $\kappa = 0$ (solid). In all cases $g_{ph} = 47\pi 10^3\,Hz$ and $\Delta = 0$ (strict resonance). Thermal distribution is expanded up to $n = 2$. Dots with error bars represent experimental data from [11]. All theoretical curves are shifted downwards by 0.05.

\[
pe_{\Delta t,\kappa}(t) = \int_0^\infty dt' p_{\Delta t}(t,t')e^{-\kappa t'}pe(t) \\
= \int_0^\infty dt' p_{\Delta t}(t,t')e^{-\kappa t'}(A + B \sin^2 \omega_R t') \\
= (1 + \kappa \Delta t)^{-t/\Delta t} \\
\times \left[A + \frac{1}{2} B \left(1 - \left[1 + \left(\frac{2\omega_R \Delta t}{1 + \kappa \Delta t}\right)^2\right]^{-1/4} \cos \left(\frac{t}{\Delta t} \arctan \frac{2\omega_R \Delta t}{1 + \kappa \Delta t}\right)\right]\right]. (4.8)
\]

The result of decoherence is therefore an exponential decay of the upper-state atomic probability together with a frequency shift. The overall damping factor $(1 + \kappa \Delta t)^{-t/\Delta t}$ is the deformed exponential [46] occurring in non-extensive thermodynamics [47], and whose links to Gamma-function averages are well known [48].

In order to get closer to experimental conditions let us suppose the system is initially described by $\rho = \rho_{atom} \otimes \rho_{field}$, where $\rho_{atom} = p_+|e\rangle\langle e| + p_-|g\rangle\langle g|$, and $\rho_{field} = \sum_{n=0}^\infty P(n)|n\rangle\langle n|$ with $P(n) = \bar{n}^n/(1 + \bar{n})(n+1)$ being a thermal distribution. The probability to find the atom in $|e\rangle$ in an ideal cavity is then described by equation

\[
pe(t) = p_+ - \left(p_+ - p_- \frac{\bar{n}}{\bar{n} + 1}\right) \sum_{n=0}^\infty P(n)g_{ph}^2(n+1) \frac{\sin^2 \sqrt{\frac{\Delta^2}{4} + g_{ph}^2(n+1)}}{\Delta^2 + g_{ph}^2(n+1)} t, \tag{4.9}
\]
Figure 4.4: Atom initially in a mixed state with $p_+ = 0.99$ interacts with the thermal field with $\bar{n} = 0.05$, described within the standard theory (dashed), and the reducible one (solid), for $N = 2 \cdot 10^4$, $Z = 0.01$. (A) parameter $\kappa = 1/440 \cdot 10^6$Hz, (B) $\kappa = 0$. All the curves correspond to $\Delta t = 0.5\mu$s. Predictions of the two theories differ by less than experimental errors. The atom-field coupling is $g_{ph} = 47\pi 10^3$Hz. Theoretical curves are shifted downwards by 0.05.

valid for the irreducible representations of CCR algebra. Applying the just described procedure, given by equation (4.8), we can evaluate the atomic behavior in non-ideal conditions. The results are depicted in figure 4.3. What follows from these plots is that the simplest model of dissipation (multiplication by an exponential factor) is unable to explain experimental data. The problem is that the axis of oscillations goes towards unity too rapidly. On the other hand, if we take randomness of the interaction time into account, theoretical predictions fit the data well only under the additional provision $\kappa = 0$, implying $T_{cav} \to \infty$. For all $\kappa > 0$ the axis of oscillations undesirably shifts towards unity (the greater $\kappa$, the faster the convergence to unity).

From the discussion in the preceding chapter, we know that predictions of the reducible representation formulation coincide formally with their irreducible counterparts in the limit of infinite number of field oscillators, $N \to \infty$. On the other hand, however, for any finite and fixed interaction time, $t$, it is still possible to find such an $N < \infty$, that predictions of the two theories remain indistinguishable. It is therefore makes sense to compare these formulations for finite values of $N$. The results are shown in figure 4.4. Plot (A) depicts comparison of the standard theory predictions (dashed) with an analogous result for the reducible representation with $N = 2 \cdot 10^4$ and $Z = 0.01$ (solid). In both cases parameter $\kappa$ is set to $\kappa = 1/440 \cdot 10^6$Hz and uncertainty of the detection time is $\Delta t = 0.5\mu$s. The two curves differ by less than experimental error bars, and it is clear that the finite-$N$ representations suffer from the same problem as the irreducible ones: Plots are shifted upwards with respect to the data points. In order to reduce this effect we need smaller $\kappa$’s. Plot (B) shows an

\footnote{For thermal fields with mean number of photons $\bar{n}$ the probability $p_e(t)$ should asymptotically approach $p_e(\infty) \to \bar{n}$, whereas $\lim_{t \to \infty} p_e(t)e^{-\kappa t} = 0$. Thus, strictly speaking, modeling dissipation by multiplication of the ideal-cavity solutions by an exponential factor is formally allowed for vacuum fields only. In order to ensure the correct behavior, we shift all the curves in figures 4.3 and 4.4 downwards by $\bar{n} = 0.05$.}
Figure 4.5: Rabi oscillations predicted within reducible representation formalism for \( N = 3000, \ Z = 1/15 \) in the system initially consisting of a mixed-state atom with \( p_+ = 0.99 \) and field in thermal state with \( \bar{n} = 0.05 \). Other parameters are \( g_{ph} = 47\pi 10^3 \) Hz, \( \kappa = 0 \) and \( \Delta t = 0.005\mu s \). (B) is a close-up of the area between the two vertical lines in (A). Notice that \( \kappa = 0 \) is unrealistic (characteristic of an ideal cavity) so that it is likely that in real atom-cavity systems other dissipative processes, which are not included in our simple model, will cause irreversible loss of energy before revival could be seen. The same effects can be caused by larger values of \( \Delta t \). In fact, setting \( \Delta t \geq 0.05\mu s \) destroys revivals seen in this figure.

In it important to verbalize explicitly the two conditions under which indistinguishability of reducible and irreducible representations, seen in figure 4.4, is valid. The first condition is the model of decoherence. As we have already seen, fitting the data needs \( \kappa = 0 \), so that the randomness of the detection time remains the only source of decoherence, making this model rather unreliable. The second condition is the interaction time limited to \( t_{eff} \approx 90\mu s \). According to the reducible formulation, it is, in principle, possible that observing the system sufficiently long a revival of vacuum oscillations will occur. This is depicted in figure 4.5. Based on these examples it is clear, however, that to be able to say anything concrete about the problem as fundamental as the choice of representation of CCR employed in field quantization, a more reliable theory of dissipation is needed. We address this issue in the next sections.

### 4.3 Phenomenological model of dissipation for \( T = 0K \)

As we have seen in the previous section the procedure of modeling dissipation by oscillation damping is inconsistent with experimental data. Here we concentrate on formal models based on master equations.
Typical approaches to dissipation from atom-cavity systems begin with Markovian master equations of the form [37, 49, 50, 51, 52]

\[
\dot{\rho} = -i[\Omega, \rho] + \gamma (\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}[\hat{a}^\dagger\hat{a}, \rho]_+),
\]  

(4.10)

where \([A, B]_+ = AB + BA\), and \(\gamma\) describes the rate of energy losses. The von Neumann part of the dynamics is governed by the Jaynes-Cummings Hamiltonian

\[
\Omega = \frac{\omega_n}{2} (|e\rangle\langle e| - |g\rangle\langle g|) + \omega\hat{a}^\dagger\hat{a} + g (\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e|).
\]

(4.11)

Creation and annihilation operators \(\hat{a}, \hat{a}^\dagger\) appearing in this equation also play a dual role of Davies jump operators [53, 54] in the dissipative part of the master equation. Following the terminology from [12] the dissipation model implied by (4.10) is referred to as the phenomenological one, in order to distinguish it from other models that will be discussed further.

Let us note already at this stage that equation (4.10) is implicitly based on a classical intuition that the system necessarily loses energy by means of losses of light, since it is light that interacts with the cavity walls, and not the atom itself (a detailed discussion of assumptions behind the phenomenological model is given in [55]). In other words, the only transitions allowed within this model are those modifying the number of photons, while keeping atomic states unchanged. As a simple illustration let us consider the atom-field system initially in \(\rho(0) = |e, 0\rangle\langle e, 0|\). For \(g = 0\) equation (4.10) implies that \(\rho(t) = \rho(0)\) for any \(t\). Even for \(g > 0\), dissipation does not occur without the prior Rabi exchange of photon to the cavity. We could say that dissipation process behaves as if it was selectively switched on (for states with non-zero photon number) and switched off (for the vacuum field state).

In principle, the dissipator in equation (4.10) could be supplemented by an additional term describing atomic spontaneous emission at its natural rate unmodified by the cavity. Within the cavity QED regime, however, this term can be neglected. In this respect, the presence of the atom inside of the cavity is practically ignored in the phenomenological model.

In the following discussion we will concentrate on dynamics that starts from the pure state \(|e, 0\rangle\langle e, 0|\). The assumption implies a convenient constraint on the Hilbert space in question. Indeed, at \(T = 0\)K the system can make transitions only downwards on the energy ladder, so that the only states that should be taken into account are: \(|1\rangle = |e, 0\rangle\), \(|2\rangle = |g, 1\rangle\) and \(|3\rangle = |g, 0\rangle\). In this basis equation (4.10) is equivalent to the following three sets of coupled first-order differential equations

\[
\begin{align*}
\dot{\rho}_{11} &= i g \rho_{12} - i g \rho_{21}, \\
\dot{\rho}_{22} &= -\gamma \rho_{22} - i g \rho_{12} + i g \rho_{21}, \\
\dot{\rho}_{12} &= -\frac{\gamma}{2} \rho_{12} + i g \rho_{11} - i g \rho_{22}, \\
\dot{\rho}_{21} &= -\frac{\gamma}{2} \rho_{21} - i g \rho_{11} + i g \rho_{22}, \\
\dot{\rho}_{33} &= \gamma \rho_{22},
\end{align*}
\]

(4.12a-e)
\[ \dot{\rho}_{13} = -i\omega_0\rho_{13} - ig\rho_{23}, \quad (4.13a) \]
\[ \dot{\rho}_{23} = -i\omega_0\rho_{23} - \frac{\gamma}{2}\rho_{23} - ig\rho_{13}, \quad (4.13b) \]

and

\[ \dot{\rho}_{31} = i\omega_0\rho_{31} + ig\rho_{32}, \quad (4.14a) \]
\[ \dot{\rho}_{32} = i\omega_0\rho_{32} - \frac{\gamma}{2}\rho_{32} + ig\rho_{31}. \quad (4.14b) \]

The loss rate \( \gamma \) does not affect the population \( \rho_{11} \), while it contributes to the decay of \( \rho_{22} \) and the increase of \( \rho_{33} \). In other words, the population of the state \( |e, 0\rangle \) is not directly affected by the dissipation, in contrast to the case of \( |g, 1\rangle \) (dissipation decreases it) and \( |g, 0\rangle \) (dissipation increases it). The coherences \( \rho_{12} \) and \( \rho_{21} \) are also damped but with the rate \( \gamma/2 \) (an interesting discussion of the dynamics of populations and coherences is given in [56]).

The initial condition implies that \( \rho_{11}(0) = 1 \), with all the other elements vanishing. Taking this into account, the last two sets of equations have solutions identically equal to zero,

\[ \rho_{13}(t) = \rho_{31}(t) = \rho_{23}(t) = \rho_{32}(t) = 0. \quad (4.15) \]

The remaining set (4.12) can be written in a matrix form

\[
\begin{pmatrix}
\rho_{11} \\
\rho_{22} \\
\rho_{12} \\
\rho_{21} \\
\rho_{33}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & ig & -ig & 0 \\
0 & -\gamma & -ig & ig & 0 \\
g & -ig & -\frac{\gamma}{2} & 0 & 0 \\
-ig & ig & 0 & -\frac{\gamma}{2} & 0 \\
0 & \gamma & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_{11} \\
\rho_{22} \\
\rho_{12} \\
\rho_{21} \\
\rho_{33}
\end{pmatrix}
\]

(4.16)

or simply

\[ \frac{d}{dt} \vec{\rho} = M\vec{\rho}. \quad (4.17) \]

Here, \( \vec{\rho} \) stands for a generalized vector whose components correspond to the elements of density matrix \( \rho \). The knowledge of eigenvectors and eigenvalues of \( M \) allows us to write it as \( M = UDU^{-1} \), where \( D \) is a diagonal matrix of its eigenvalues and \( U \) is composed of eigenvectors arranged in columns. Making use of this equation, formula (4.17) can be recast as

\[ \frac{d}{dt} \vec{\rho} = UD\bar{U}^{-1}\vec{\rho}, \quad (4.18) \]
whose formal solution is

\[ \rho(t) = e^{UDU^{-1}} \rho(0) = U e^{Di} U^{-1} \rho(0), \quad \rho(0) = (1, 0, 0, 0)^T. \] (4.19)

The elements of \( \rho(t) \) are

\[
\begin{align*}
\rho_{11}(t) &= -\frac{8g^2}{\gamma^2 - 16g^2} e^{-\gamma t} \left( \frac{\gamma}{2} \alpha t \sqrt{\gamma^2 - 16g^2} \right) + \frac{\gamma^2 - 8g^2}{2(\gamma^2 - 16g^2)} e^{-\gamma t} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} + e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right), \\
\rho_{22}(t) &= \frac{4g^2}{\gamma^2 - 16g^2} e^{-\gamma t} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} + e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right) - \frac{8g^2}{\gamma^2 - 16g^2} e^{-\gamma t}, \\
\rho_{33}(t) &= 1 + \frac{16g^2}{\gamma^2 - 16g^2} e^{-\gamma t} \left( \frac{\gamma^3}{2(\gamma^2 - 16g^2)^{3/2}} \alpha t \sqrt{\gamma^2 - 16g^2} \right) - \frac{\gamma^2}{2(\gamma^2 - 16g^2)} \alpha t \sqrt{\gamma^2 - 16g^2} + \frac{8g^2 \gamma}{(\gamma^2 - 16g^2)^{3/2}} e^{-\gamma t} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} - e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right), \\
\rho_{12}(t) &= -2i \frac{g \gamma}{\gamma^2 - 16g^2} e^{-\gamma t} + i \frac{g}{\sqrt{\gamma^2 - 16g^2}} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} - e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right) e^{-\gamma t} + i \frac{g \gamma}{\gamma^2 - 16g^2} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} + e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right) e^{-\gamma t}, \\
\rho_{21}(t) &= 2i \frac{g \gamma}{\gamma^2 - 16g^2} e^{-\gamma t} - i \frac{g}{\sqrt{\gamma^2 - 16g^2}} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} - e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right) e^{-\gamma t} - i \frac{g \gamma}{\gamma^2 - 16g^2} \left( e^{\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} + e^{-\frac{1}{2} \alpha t \sqrt{\gamma^2 - 16g^2}} \right) e^{-\gamma t}.
\end{align*}
\] (4.20a, 4.20b, 4.20c, 4.20d, 4.20e)

We will be particularly interested in the probabilities \( p_{e,0}^{ph}(t) = \rho_{11}(t) \), \( p_{g,1}^{ph}(t) = \rho_{22}(t) \), and \( p_{g,0}^{ph}(t) = \rho_{33}(t) \) (“ph” is an abbreviation for “phenomenological”). The latter two probabilities can be interpreted as corresponding to the photon inside and outside of the system, respectively. Noting that quantity \( \gamma^2 - 16g^2 \) appearing in all the above equations takes negative values within the strong coupling regime (\( g \) is much larger than atomic and cavity relaxation times) we may write
Figure 4.6: Predictions of the phenomenological model for $T = 0\text{K}$ (solid) and $T = 0.8\text{K}$ (dashed) for $g = 47\pi 10^3\text{Hz}$. The dissipation rate for $T = 0$ is $\gamma = 0.3g$, while $\gamma_1 = 0.3g$ and $\gamma_\perp = 0.0466\gamma$ for $T = 0.8\text{K}$. Plot (A) shows the probability of finding the atom in its ground state, $p_{g,0}^\text{ph}(t) = p_{g,0}^\text{ph}(t) + p_{g,1}^\text{ph}(t)$, (B) shows the probability for a photon to be outside of the cavity, $p_{g,0}^\text{ph}(t)$.

\begin{align}
 p_{e,0}^\text{ph}(t) &= \frac{8g^2}{16g^2 - \gamma^2} e^{-\gamma t} + \frac{\gamma}{\sqrt{16g^2 - \gamma^2}} e^{-\gamma t} \sin \frac{t}{2} \sqrt{16g^2 - \gamma^2} \\
 &\quad + \frac{8g^2 - \gamma^2}{16g^2 - \gamma^2} e^{-\gamma t} \cos \frac{t}{2} \sqrt{16g^2 - \gamma^2}, \quad (4.21a) \\
 p_{g,1}^\text{ph}(t) &= \frac{8g^2}{16g^2 - \gamma^2} e^{-\gamma t} - \frac{8g^2}{16g^2 - \gamma^2} e^{-\gamma t} \cos \frac{t}{2} \sqrt{16g^2 - \gamma^2}, \quad (4.21b) \\
 p_{g,0}^\text{ph}(t) &= 1 - \frac{16g^2}{16g^2 - \gamma^2} e^{-\gamma t} + \frac{\gamma^3}{\left(16g^2 - \gamma^2\right)^{3/2}} e^{-\gamma t} \sin \frac{t}{2} \sqrt{16g^2 - \gamma^2} \\
 &\quad + \frac{\gamma}{16g^2 - \gamma^2} e^{-\gamma t} \cos \frac{t}{2} \sqrt{16g^2 - \gamma^2} - \frac{16g^2\gamma}{\left(16g^2 - \gamma^2\right)^{3/2}} e^{-\gamma t} \sin \frac{t}{2} \sqrt{16g^2 - \gamma^2}. \quad (4.21c)
\end{align}

Asymptotically, for $t \to \infty$, these probabilities are $\lim_{t \to \infty} p_{g,0}^\text{ph}(t) = 1$ and $\lim_{t \to \infty} p_{g,1}^\text{ph}(t) = \lim_{t \to \infty} p_{g,0}^\text{ph}(t) = 0$, reflecting the fact that the photon must finally escape from the cavity. An example of the evolution of $p_g^\text{ph}(t) = p_{g,0}^\text{ph}(t) + p_{g,1}^\text{ph}(t)$ and $p_{g,0}^\text{ph}(t)$ is shown in figure 4.6 (solid curves). Although the damping parameter $\gamma$ is chosen arbitrarily, the qualitative properties of the dynamics are representative for the phenomenological model in general.

### 4.4 Phenomenological model at $T > 0\text{K}$. First approximation

Consider now a more complex but at the same time more realistic case where the environment is at non-zero temperature. In that case, phenomenological master equation (4.10) acquires
an additional term related to upwards transitions on the energy ladder,

\[ \frac{d\rho}{dt} = -i[H, \rho] + \gamma_1 \left( \hat{a} \rho \hat{a}^\dagger - \frac{1}{2} [\hat{a}^\dagger \hat{a}, \rho]_+ \right) + \gamma_\uparrow \left( \hat{a}^\dagger \rho \hat{a} - \frac{1}{2} [\hat{a} \hat{a}^\dagger, \rho]_+ \right), \]  

(4.22)

where \( \gamma_1 \) and \( \gamma_\uparrow \) denote rates of transitions downwards and upwards, respectively. Given temperature \( T \), these two coefficients are related by the formula (a consequence of the KMS condition [38])

\[ \gamma_\uparrow = e^{-\frac{\hbar \omega_0}{2kT}} \gamma_1. \]  

(4.23)

Inserting \( T \) and \( \omega_0 \) as reported in the experiment [11] \( (T = 0.8K \text{ and } \omega_0 = 2\pi 51.099 \text{GHz}) \) yields

\[ \gamma_\uparrow = \epsilon \gamma_1, \quad \text{with } \epsilon \approx 0.0466327, \]  

(4.24)

which, in fact, means that equation (4.22) contains a single parameter \( \gamma_1 \).

The most important formal difference with respect to \( T = 0K \) is that now the number of states that take part in the evolution becomes infinite. This is a consequence of two facts. First, the form of the thermal density matrix at \( t = 0 \), being a mixture of \( n \)-photon states with \( n = 0, 1, \ldots, \infty \), and second, the possibility of upwards transitions implied by \( \gamma_\uparrow \neq 0 \). However, for \( T = 0.8K \) the contribution to the initial condition from all non-vacuum states is not greater than 5% of probability, and the upwards transitions out of our three-dimensional Hilbert space are of the same order. This means that although the number of excitations can, in principle, increase in time, the probability to find more than one is less then 0.05. So the technical assumption that can be made here is that in the first approximation it is legitimate to truncate the Hilbert space to its three-dimensional subspace spanned by the already mentioned states \( |1\rangle = |e, 0\rangle, |2\rangle = |g, 1\rangle \) and \( |3\rangle = |g, 0\rangle \) and, additionally, include the upwards transitions within this subspace. The next approximation would be to include the subspace spanned by \( |e, 1\rangle \) and \( |g, 2\rangle \) (two excitations in total) with downwards transitions.

Restriction of the Hilbert space requires modification of the form of Davies operators to guarantee that higher dimensional subspaces of the Hilbert space will not couple to the subspace in question. Let \( \Pi \)

\[ \Pi = |e, 0\rangle \langle e, 0| + |g, 1\rangle \langle g, 1| + |g, 0\rangle \langle g, 0|, \]  

(4.25)

be the operator projecting on the subspace of interest. Then, the truncated Davies operators are obtained by the substitution

\[ \hat{a} \rightarrow \Pi \hat{a} \Pi = |g, 0\rangle \langle g, 1|, \]  

(4.26)

\[ \hat{a}^\dagger \rightarrow \Pi \hat{a}^\dagger \Pi = |g, 1\rangle \langle g, 0|, \]  

(4.27)

that modifies master equation (4.22) to the form
\[
\dot{\rho} = -i[\Omega, \rho] + \gamma_1 (|g, 0\rangle\langle g, 1| \rho|g, 1\rangle\langle g, 0| - \frac{1}{2} [|g, 1\rangle\langle g, 1|, \rho]_+) + \gamma_1 (|g, 1\rangle\langle g, 0| \rho|g, 0\rangle\langle g, 1| - \frac{1}{2} [|g, 0\rangle\langle g, 0|, \rho]_+).
\]
(4.28)

This explicitly shows that the dissipative term again does not affect the state \(|e, 0\rangle\) of the system.

Matrix elements of \(\dot{\rho}\), equation (4.28), between basis states split into three groups of equations:

\[
\begin{align*}
\dot{\rho}_{11} &= i g \rho_{12} - i g \rho_{21}, \\
\dot{\rho}_{22} &= -\gamma_1 \rho_{22} - i g \rho_{12} + i g \rho_{21} + \gamma_1 \rho_{33}, \\
\dot{\rho}_{12} &= -\frac{\gamma_1}{2} \rho_{12} + i g \rho_{11} - i g \rho_{22}, \\
\dot{\rho}_{21} &= -\frac{\gamma_1}{2} \rho_{21} - i g \rho_{11} + i g \rho_{22}, \\
\dot{\rho}_{33} &= \gamma_1 \rho_{22} - \gamma_1 \rho_{33},
\end{align*}
\]
(4.29a-d)

\[
\begin{align*}
\dot{\rho}_{13} &= -i \omega_0 \rho_{13} - i g \rho_{23} - \frac{\gamma_1}{2} \rho_{13}, \\
\dot{\rho}_{23} &= -i \omega_0 \rho_{23} - i g \rho_{13} - \frac{\gamma_1 + \gamma_1}{2} \rho_{23},
\end{align*}
\]
(4.30a-b)

and

\[
\begin{align*}
\dot{\rho}_{31} &= i \omega_0 \rho_{31} + i g \rho_{32} - \frac{\gamma_1}{2} \rho_{31}, \\
\dot{\rho}_{32} &= i \omega_0 \rho_{32} + i g \rho_{31} - \frac{\gamma_1 + \gamma_1}{2} \rho_{32}.
\end{align*}
\]
(4.31a-b)

It is obvious that for \(\gamma_1 = 0\) we recover the situation characteristic of \(T = 0\)K and the equations reduce to (4.12)-(4.14). Things complicate, however, for \(\gamma_1 > 0\). Assuming that the atom-field system starts from \(\rho(0) = |e, 0\rangle\langle e, 0|\) we find that (4.30) and (4.31) again have trivial solutions \(\rho_{13}(t) = \rho_{31}(t) = \rho_{32}(t) = 0\), so that it is sufficient to focus attention on equations (4.29). Unfortunately, although the number of equations is the same as for \(T = 0\)K, we are not able to determine analytically the eigenvectors of the matrix of coefficients appearing at the right hand side of (4.29). Numerical analysis shows, however, that predictions are basically the same as their counterparts for \(T = 0\)K, as shown in figure 4.6 (dashed curves). Figure 4.7, on the other hand, depicts the appropriate probabilities for several different \(\epsilon\)s. The range of temperatures we used makes this exercise purely formal, but it clearly shows the tendency for lowering the curves as \(\epsilon\) grows.
Figure 4.7: Predictions of the phenomenological model for $\epsilon = 0.01$ (solid), $\epsilon = 0.1$ (dashed), $\epsilon = 0.5$ (dot-dashed) and $\epsilon = 1$ (dotted) and $g = 47\pi 10^3 \text{Hz}$, $\gamma_\perp = 0.3g$. The corresponding temperatures are, respectively, $T \approx 0.54 \text{K}$, $T \approx 1.1 \text{K}$, $T \approx 3.5 \text{K}$ and $T = \infty$. As mentioned in the text, this illustration should serve to get some intuitions about the phenomenological model, as the restriction of the Hilbert space to the three dimensional subspace is a gross abuse here (the greater temperature the greater probability of the states with large photon numbers).

4.5 Dressed-state basis description

Relatively recently Scala et al. [12, 13] questioned the very basics of the phenomenological approach to dissipation in atom-field systems, and proposed an apparently cosmetic modification of it (we will refer to it as a microscopic model). The idea goes back to the original concepts of Davies on how to describe system-reservoir interactions in Markovian master equations [53, 54]. The modification is essential for general consistency of the dynamics: The jump operators should describe jumps between eigenstates of the system Hamiltonian $H$ (i.e. dressed states) [57], and not eigenstates of the free-field Hamiltonian $\hbar \omega \hat{a}^\dagger \hat{a}$, as it happens in practically all approaches employed in quantum optics [58, 59]. The main objection to the phenomenological master equation relates therefore to the set of elementary transformations to which the reduced system is subjected during the interaction with its environment. From a formal point of view, the essence of microscopic treatments of dissipation is in the way the Davies operators are constructed. According to [12] these operators are defined as

$$A(\Omega_{M',l} - \Omega_{M,m}) = |\Omega_{M,m}\rangle \langle \Omega_{M,M'} |(\hat{a} + \hat{a}^\dagger)\langle \Omega_{M',l}|$$

(4.32)

$$= \frac{1}{2} \delta_{M,M'-1}(\sqrt{M+1} + lm\sqrt{M}) |\Omega_{M,m}\rangle \langle \Omega_{M+1,l}|,$$  

(4.33)

where $l, m = \pm$ and $M$ refers to the total number of excitations in the atom-cavity system. The sum of annihilation and creation operators in the first line of this equation appears due to the implicit form of the system-environment interaction Hamiltonian, which is taken to be
\( (\hat{a} + \hat{a}^\dagger) \otimes B \). The idea underlying this equation does not change the system itself but only the way it interacts with the reservoir. One feature, to which we shall refer in the following, is that operators derived this way can describe transitions between neighboring manifolds of dressed states, but not within a single manifold. Indeed, in the basis \( |1\rangle = |\Omega_+\rangle, |2\rangle = |\Omega_-\rangle \) and \( |3\rangle = |\Omega_0\rangle \), \( M, M' \in \{0, 1\} \), the only operators implied by (4.33) are \( |\Omega_0\rangle \langle \Omega_\pm| \). So the model excludes, in particular, transitions between \( |\Omega_+\rangle \) and \( |\Omega_-\rangle \). This makes sense in closed cavities since the wavelength of an appropriate transition is of the order of kilometers, while the cavity sizes are much smaller (a few centimeters). It is not, however, the case for open cavities, where long-wave radiation plays a role of a constant noise. A simple argument to support proposition of Scala et al. is that interacting atoms and photons are as inseparable as protons and electrons in atoms and — similarly to atoms — should be assumed to loose energy via transitions between appropriate energy levels.

At \( T = 0K \) the master equation proposed in [12] reads

\[
\dot{\rho} = -i[\Omega, \rho] + \gamma_1 \left( \frac{1}{2} |\Omega_0\rangle \langle \Omega_+| \rho |\Omega_0\rangle \langle \Omega_+| - \frac{1}{4} |\Omega_0\rangle \langle \Omega_+| \rho |\Omega_+\rangle \langle \Omega_0| \right) + \gamma_2 \left( \frac{1}{2} |\Omega_0\rangle \langle \Omega_-| \rho |\Omega_0\rangle \langle \Omega_-| - \frac{1}{4} |\Omega_0\rangle \langle \Omega_-| \rho |\Omega_-\rangle \langle \Omega_0| \right),
\]

(4.34)

where \( |\Omega_\pm\rangle = 1/\sqrt{2}(|g, 1\rangle \pm |e, 0\rangle) \) are the dressed atomic states, \( |\Omega_0\rangle = |g, 0\rangle \), while \( \gamma_1 \) and \( \gamma_2 \) denote rates at which the corresponding transitions \( |\Omega_\pm\rangle \to |\Omega_0\rangle \) take place.

When comparing this equation with (4.10) or (4.22) we see that it contains two jump operators, \( |\Omega_0\rangle \langle \Omega_+| \) and \( |\Omega_0\rangle \langle \Omega_-| \) expressed in terms of the dressed states, and both of them describe transitions towards the system ground state \( |\Omega_0\rangle = |g, 0\rangle \). Within this model, therefore, both bare states \( |g, 1\rangle \) and \( |e, 0\rangle \) may decay towards \( |g, 0\rangle \) contributing to the irreversible loss of energy from the system. In fact, since the description is now in terms of the dressed states, being combinations of the bare ones, even a single decay channel, \( |\Omega_+\rangle \to |\Omega_0\rangle \) or \( |\Omega_-\rangle \to |\Omega_0\rangle \), produces decays of the two non-ground bare states, \( |e, 0\rangle \) and \( |g, 1\rangle \). It is worthy of noting, that the essence of the phenomenological model restricted to the three-dimensional Hilbert space in question is that the only state that may directly decay is \( |g, 1\rangle \).

Assume that the initial state of the atom-cavity system is \( |e, 0\rangle \), then master equation (4.34) takes the form
The corresponding probabilities of the bare states are
\[ p_{9,0}^{\text{mic}}(t) = 1 - \frac{1}{2} e^{-\frac{\gamma_1 t}{2}} - \frac{1}{2} e^{-\frac{\gamma_2 t}{2}}, \]  
\[ p_{9,1}^{\text{mic}}(t) = \frac{1}{4} e^{-\frac{\gamma_1 t}{2}} + \frac{1}{4} e^{-\frac{\gamma_2 t}{2}} - \frac{1}{2} e^{-\frac{\gamma_1 + \gamma_2 t}{2}} \cos 2gt. \]

which is in fact simpler than the ones corresponding to the phenomenological model. Its solution is

\[ \rho_{11}(t) = e^{-\frac{\gamma_1 t}{2}} \rho_{11}(0), \]  
\[ \rho_{22}(t) = e^{-\frac{\gamma_2 t}{2}} \rho_{22}(0), \]  
\[ \rho_{33}(t) = \left(1 - e^{-\frac{\gamma_1 t}{2}}\right) \rho_{11}(0) + \left(1 - e^{-\frac{\gamma_2 t}{2}}\right) \rho_{22}(0) + \rho_{33}(0), \]  
\[ \rho_{12}(t) = e^{(-i(\Omega_+ - \Omega_-) - \frac{\gamma_1 + \gamma_2}{4}) t} \rho_{12}(0), \]  
\[ \rho_{13}(t) = e^{(-i(\Omega_+ - \Omega_0) - \frac{\gamma_1}{4}) t} \rho_{13}(0), \]  
\[ \rho_{23}(t) = e^{(-i(\Omega_- - \Omega_0) - \frac{\gamma_2}{4}) t} \rho_{23}(0), \]

where \( \Omega_+ - \Omega_- = 2g, \Omega_+ - \Omega_0 = \omega_0 \pm g \). In exact resonance the initial condition satisfies

\[ \rho(0) = \langle e, 0 \rangle \langle e, 0 \rangle = \frac{1}{2} \langle \Omega_+ \rangle \langle \Omega_+ \rangle + \frac{1}{2} \langle \Omega_- \rangle \langle \Omega_- \rangle - \frac{1}{2} \langle \Omega_+ \rangle \langle \Omega_- \rangle - \frac{1}{2} \langle \Omega_- \rangle \langle \Omega_+ \rangle, \]  

and after some algebra it can be shown that

\[ \rho(t) = \frac{1}{2} e^{-\frac{\gamma_1 t}{2}} |\Omega_+\rangle \langle \Omega_+ | + \frac{1}{2} e^{-\frac{\gamma_2 t}{2}} |\Omega_-\rangle \langle \Omega_- | + \left(1 - \frac{1}{2} e^{-\frac{\gamma_1 t}{2}} - \frac{1}{2} e^{-\frac{\gamma_2 t}{2}}\right) |\Omega_0\rangle \langle \Omega_0 | \]
\[ -\frac{1}{2} e^{2igt} e^{-\frac{\gamma_1 + \gamma_2 t}{4}} |\Omega_-\rangle \langle \Omega_+ | - \frac{1}{2} e^{-2igt} e^{-\frac{\gamma_1 + \gamma_2 t}{4}} |\Omega_+\rangle \langle \Omega_- |. \]  

The corresponding probabilities of the bare states are

\[ p_{9,0}^{\text{mic}}(t) = 1 - \frac{1}{2} e^{-\frac{\gamma_1 t}{2}} - \frac{1}{2} e^{-\frac{\gamma_2 t}{2}}, \]  
\[ p_{9,1}^{\text{mic}}(t) = \frac{1}{4} e^{-\frac{\gamma_1 t}{2}} + \frac{1}{4} e^{-\frac{\gamma_2 t}{2}} - \frac{1}{2} e^{-\frac{\gamma_1 + \gamma_2 t}{2}} \cos 2gt. \]
Figure 4.8: Predictions of the microscopic approach. Both plots show the probability of finding the atom in its ground state under different conditions: $g = 47\pi 10^3$Hz; (A) $\gamma_1 = 0.1g$, $\gamma_2 = 0$, and (B) $\gamma_1 = 0.1g$, $\gamma_2 = 0.05g$.

The probability $p_{g}^{\text{mic}}(t)$ of finding the atom in the lower state,

$$p_{g}^{\text{mic}}(t) = p_{g,0}^{\text{mic}}(t) + p_{g,1}^{\text{mic}}(t) = 1 - \frac{1}{4}e^{-\gamma_1 t} - \frac{1}{4}e^{-\gamma_2 t} - \frac{1}{2}e^{-\gamma_1 + \gamma_2 t} \cos 2gt,$$

(4.41)

is symmetric with respect to $\gamma_1$ and $\gamma_2$, and it is the sum of these coefficients that controls the rate at which oscillations are damped. An interesting situation (population trapping) takes place when one sets $\gamma_1 = 0$ or $\gamma_2 = 0$. In this case, the probability of finding the atom in the lower state approaches asymptotically $3/4$ instead of $1$. In contrast, for non-zero $\gamma_1$ and $\gamma_2$, the limiting value is $1$, as shown in figure 4.8.

The following sections confront the above two models with the experiment reported in [11], and then propose yet another modification of the master equation, appropriate for an open cavity.

### 4.6 Role of Gaussian profile of the cavity field

The above calculations were performed under the assumption that the atom-field coupling $g$ is constant in time. Still, in real experiments the coupling is effectively time dependent since the atoms are moving through the cavity. A convenient formulation of the problem is to work in the atomic rest frame, where the moving cavity field can be modeled by a time-dependent coupling parameter $g(t)$ [33]. At first, we examine the consequences of exploiting the time-dependent atom-field coupling for the microscopic model of dissipation, as it turns out to be far more simple than for the phenomenological one.

Let us begin with equation (4.34) written as $\dot{\rho}(t) = \mathcal{L}(g)\rho(t)$. Its formal solution is then

$$\rho(t) = e^{\mathcal{L}(g)(t-t_0)}\rho(t_0).$$

(4.42)
The interaction time is split here into \( n = 21 \) equally-sized time intervals of length \( \Delta t = t/21 \). \( g(t) \) is expressed in units of \( g \) and units of time are arbitrary. The distance between the two vertical lines in the central part of the plot is \( 2w \). The shape of the Gaussian corresponds to the mode profile inside of the cavity discussed in [11]. In the actual numerical simulation of the phenomenological model \( n = 20001 \).

One way to deal with the time-dependent coupling is to take \( g(t) \) as a discrete \( n \)-step approximation of the Gaussian \( g e^{-(vt)^2/w^2} \), as shown in figure 4.9. This allows us to consider dynamics of the system in non-overlapping intervals \( \Delta t = t/n \), in each of which the atom-field coupling is constant, that is \( g(t_j) = g_j, t_j \in [(j-1)\Delta t, j\Delta t], j \in \{1, \ldots, n\} \). With this in mind, equation (4.42) can be recast as

\[
\rho(t) = e^{L(g_n)\Delta t} \cdots e^{L(g_1)\Delta t} \rho(0). \tag{4.43}
\]

In exact resonance, the basis of dressed states is independent of \( g_j \). Under this assumption, the diagonal elements of \( \rho(t) \) given by equations (4.36a)-(4.36c) remain valid, while the dependence of \( \rho(t) \) on coupling constants is exclusively via exponents of the off-diagonal element \( \rho_{12}(t) \),

\[
\rho_{12}(t) = e^{-\left(2i g_n + \frac{\gamma_1}{4} + \frac{\gamma_2}{4}\right)\Delta t} \cdots e^{-\left(2i g_1 + \frac{\gamma_1}{4} + \frac{\gamma_2}{4}\right)\Delta t} \rho_{12}(0)
\]

\[
= e^{-\frac{1}{4} \Delta t} e^{-2i \int \Delta t} \rho_{12}(0), \tag{4.44}
\]

where \( \rho_{12}(0) = -1/2 \). For \( \Delta t \to 0 \) the sum of products \( g_i \Delta t \) can be replaced by an integral over the interaction time

\[
g_1 \Delta t + \cdots g_n \Delta t \approx \int_0^t dt' g(t') = g \int_0^t dt' e^{-\frac{(vt')^2}{w^2}} \approx g \sqrt{\frac{\pi}{v}} w. \tag{4.45}
\]

This allows us to compute the explicit formula for \( \rho(t) \) valid for the Gaussian,

\[
\rho(t) = \begin{pmatrix}
\frac{1}{2} e^{-\frac{\gamma_1}{4} t} & -\frac{1}{2} e^{-\frac{1}{4} \Delta t} e^{-2i g \sqrt{\frac{\pi}{v}} t} & 0 \\
-\frac{1}{2} e^{-\frac{1}{4} \Delta t} e^{2i g \sqrt{\frac{\pi}{v}} t} & \frac{1}{2} e^{-\frac{\gamma_1}{4} t} & 0 \\
0 & 0 & 1 - \frac{1}{2} e^{-\frac{\gamma_1}{4} t} - \frac{1}{2} e^{-\frac{\gamma_2}{4} t}
\end{pmatrix}, \tag{4.46}
\]

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Figure 4.10: Predictions of the microscopic model at $T = 0$, (A) with and (B) without the Gaussian structure of the coupling. In (A) the atom most of the time experiences coupling much weaker from the maximal value. Parameters are $g = 47\pi 10^3$ Hz, $\gamma_1 = 0.12g$, and $\gamma_2 = 0.01g$. $t$ is the true time.

by means of the limit $n \to \infty$. The coupling parameter $g$ gets indeed replaced by $g_{\text{eff}} = g\sqrt{\pi w/d}$, as suggested in [11], but the exponents involving $\gamma_1$ and $\gamma_2$ remain unchanged (see discussion in section 4.1). Based on this equation, the probability of the atomic ground state is

$$p_{g}^{\text{mic}}(t) = 1 - \frac{1}{4}e^{-\frac{\gamma_2}{2}t} - \frac{1}{4}e^{-\frac{\gamma_1}{2}t} - \frac{1}{2}e^{-\frac{\gamma_1 + \gamma_2}{4}t}\cos 2g\sqrt{\pi w}\frac{w}{d}t,$$  \hspace{1cm} (4.47)

and the only modification with respect to (4.41) is in the argument of the cosine function.

Figure 4.10A shows how the effective Rabi frequency obtained for a Gaussian coupling differs from the one found for a constant $g = \max_t \{g(t)\}$, depicted in figure 4.10B. It is evident that one of the consequences of the non-trivial mode structure is the slowing down of the coherent evolution in the atom-cavity system.

The case of the phenomenological model turns out to be more complicated, not only because at $T > 0$K we did not manage to solve analytically the underlying eigenvalue equation, but also because even at $T = 0$K the coupling constants enter the solutions (4.20) in a very complicated way. So even at zero temperature we would have to rely on numerical analysis. The case $T = 0.8$K is shown in figure 4.11B. Having no access to the closed-form solution of the master equation we do not know how to switch between true and effective times. This is why this figure depicts experimental data rescaled to the true time ($0 \leq t \leq 430\mu s$ of the true time corresponds to $0 \leq t_{\text{eff}} \leq 91\mu s$ of the effective one).

Disagreement between theory and experiment is here at least as bad as in the microscopic model. The structure of solutions of these two models does not allow for fitting the data by any choice of the parameters.
Figure 4.11: (A) Predictions of the microscopic model with the Gaussian coupling at \( T = 0 \) as a function of \( t_{\text{eff}} \). The parameters are \( \gamma_1 = 0.12g \) and \( \gamma_2 = 0.01g \). (B) Predictions of the phenomenological model with the \( n \)-step approximation \( (n = 20001) \) of the Gaussian coupling at \( T = 0.8K \), \( w = 5.96mm \), \( g = 47\pi10^5Hz \). \( \Delta t = t/20001 \) for each time point (for \( 0 \leq t \leq 430\mu s \) with \( 1\mu s \) step). \( t \) is the true time. The effective Rabi frequency is somewhat smaller than \( g_{\text{eff}} \) implied by the microscopic model.

Figure 4.12: Energy levels and decay coefficients used in the generalized model; \( E_{\pm} = \hbar\Omega_{\pm} \), \( E_0 = \hbar\Omega_0 \).

4.7 Open-cavity generalization of the Scala et al. model

At \( T = 0.8K \) the average number of photons with energy \( \Omega_+ - \Omega_- = 2g \) is \( \bar{n} = 354666 \), and the corresponding wavelength is 6378.56m. If the 5cm cavity of Brune et al. was a closed one, there would be no reason to take \( 2g \) transitions into account. However, in open cavities [32, 60] such large-wavelength oscillations cannot be neglected and play a role of a constant noise. An instructive exercise is to compute the temperature that would be needed to produce \( \bar{n} = 354666 \) if one took, instead of \( 2g \), the resonant frequency \( \omega_0 \). The result is \( T = 869770K \). The KMS condition then implies that the systems jumps up and down between the two states with practically identical rates, \( \gamma_\downarrow \approx \gamma_\uparrow \). Although in all further considerations the initial thermal noise is, in fact, neglected, the paradigm of an open cavity interaction permits for a situation where the system can lose and then gain energy by additional transitions between the dressed states within a given manifold. The question we should now deal with is not whether these low-energetic transitions should be
taken into account but rather to what extent they do affect the evolution of the system.

In order to take this process into account the microscopic master equation (4.34) should be supplemented with additional Davies operator \(|\Omega_-\rangle\langle\Omega_+|\). As has been suggested by Alicki, [57], this can be done formally by consideration of a specific form of a system-environment interaction Hamiltonian with an additional term proportional to \(\hat{a}^{\dagger}\hat{a}\) (the implicit assumption in both phenomenological and microscopic models is that interaction with environment is linear in field operators, \(\hat{a}\) and \(\hat{a}^{\dagger}\)). With the common assumption about the Hilbert space restricted to the three-dimensional subspace spanned by vectors \(|1\rangle = |\Omega_+\rangle\), \(|2\rangle = |\Omega_-\rangle\) and \(|3\rangle = |\Omega_0\rangle\), the generalized master equation for \(T = 0\K\) can be written as follows

\[
\dot{\rho} = -i[\Omega, \rho] + \gamma_1 \left\{ \frac{1}{2} |\Omega_0\rangle \langle \Omega_+| \rho |\Omega_0\rangle \langle \Omega_+| - \frac{1}{4} |\Omega_+\rangle \langle \Omega_+|, \rho \right\}_+ + \gamma_2 \left\{ \frac{1}{2} |\Omega_0\rangle \langle \Omega_-| \rho |\Omega_0\rangle \langle \Omega_-| - \frac{1}{4} |\Omega_-\rangle \langle \Omega_-|, \rho \right\}_+ + \gamma_3 \left\{ \frac{1}{2} |\Omega_-\rangle \langle \Omega_+| \rho |\Omega_-\rangle \langle \Omega_+| - \frac{1}{4} |\Omega_+\rangle \langle \Omega_+|, \rho \right\}_+ ,
\]

(4.48)

where the meaning of decay coefficients follows from figure 4.12 (the derivation of this equation from first principles is shown in Appendix). It is obvious that \(\gamma_3 = 0\) recovers the microscopic master equation. It must be remembered, however, that \(|\Omega_-\rangle\langle\Omega_+|\) cannot be derived within the microscopic model based on jump operators defined by equation (4.33).

To solve (4.48) we use the damping basis method [61]. It is based on the observation that the knowledge of right eigenoperators of \(L\), \(\dot{\rho} = L\rho\), allows for writing the formal solution for \(\rho\) as

\[
\rho(t) = e^{Lt}\rho(0) = e^{Lt} \sum_i c_i \rho_i = \sum_i c_i \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n \rho_i = \sum_i c_i \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda_i^n \rho_i = \sum_i c_i e^{\Lambda_i t} \rho_i ,
\]

(4.49)

where \(L\rho_i = \Lambda_i \rho_i\) and \(\rho(0) = \sum_i c_i \rho_i\). Omitting exact calculation, it can be shown that eigenoperators and eigenvalues of \(L\), implied by equation (4.48), are
\begin{align}
\rho_1 &= - (\gamma_1 - \gamma_2 + \gamma_3) |\Omega_+\rangle \langle \Omega_+| + \gamma_3 |\Omega_0\rangle \langle \Omega_0| + (\gamma_1 - \gamma_2) |\Omega_0\rangle \langle \Omega_0|, \\
\Lambda_1 &= - \frac{\gamma_1 + \gamma_3}{2}, \quad (4.50a) \\
\rho_2 &= |\Omega_-\rangle \langle \Omega_-| - |\Omega_0\rangle \langle \Omega_0|, \quad \Lambda_2 = - \frac{\gamma_2}{2}, \quad (4.50b) \\
\rho_3 &= (\gamma_1 \gamma_2 + \gamma_2 \gamma_3) |\Omega_0\rangle \langle \Omega_0|, \quad \Lambda_3 = 0, \quad (4.50c) \\
\rho_4 &= |\Omega_+\rangle \langle \Omega_-|, \quad \Lambda_4 = - i (\Omega_+ - \Omega_-) - \frac{\gamma_1 + \gamma_2 + \gamma_3}{4}, \quad (4.50d) \\
\rho_5 &= |\Omega_+\rangle \langle \Omega_0|, \quad \Lambda_5 = - i (\Omega_+ - \Omega_0) - \frac{\gamma_1 + \gamma_3}{4}, \quad (4.50e) \\
\rho_6 &= |\Omega_-\rangle \langle \Omega_0|, \quad \Lambda_6 = - i (\Omega_- - \Omega_0) - \frac{\gamma_2}{4}, \quad (4.50f) \\
\rho_7 &= |\Omega_-\rangle \langle \Omega_+|, \quad \Lambda_7 = i (\Omega_+ - \Omega_-) - \frac{\gamma_1 + \gamma_2 + \gamma_3}{4}, \quad (4.50g) \\
\rho_8 &= |\Omega_0\rangle \langle \Omega_+|, \quad \Lambda_8 = i (\Omega_+ - \Omega_0) - \frac{\gamma_1 + \gamma_3}{4}, \quad (4.50h) \\
\rho_9 &= |\Omega_0\rangle \langle \Omega_-|, \quad \Lambda_9 = i (\Omega_- - \Omega_0) - \frac{\gamma_2}{4}. \quad (4.50i)
\end{align}

This set constitutes a basis of operators defined on the Hilbert space of the system in question. The initial state \( \rho(0) = |e, 0\rangle \langle e, 0| \) can be now recast as

\[
\rho(0) = - \frac{1}{2} \frac{1}{\gamma_1 - \gamma_2 + \gamma_3} \rho_1 + \frac{1}{2} \frac{\gamma_1 - \gamma_2 + 2 \gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} \rho_2 + \frac{1}{\gamma_2 (\gamma_1 + \gamma_3)} \rho_3 - \frac{1}{2} \rho_4 - \frac{1}{2} \rho_7. \quad (4.51)
\]

This is the point, where the major advantage of the damping basis method comes into play: It is sufficient to multiply all \( \rho_i \) by the corresponding exponentials \( e^{\Lambda_i t} \) to get \( \rho(t) \) for any \( t \),

\[
\rho(t) = - \frac{1}{2} \frac{1}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{1}{2} + \frac{\gamma_3}{2} t} \rho_1 + \frac{1}{2} \frac{\gamma_1 - \gamma_2 + 2 \gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{2}{2} t} \rho_2 + \frac{1}{\gamma_2 (\gamma_1 + \gamma_3)} \rho_3 - \frac{1}{2} e^{2\gamma_3 t} e^{-\frac{1}{2} + \frac{\gamma_2 + \gamma_3}{4} t} \rho_4 - \frac{1}{2} e^{2\gamma_1 t} e^{-\frac{1}{2} + \frac{\gamma_2 + \gamma_3}{4} t} \rho_7. \quad (4.52)
\]

The evolution of the lower atomic-state population is now of the form

\[
p_g(t) = 1 - \frac{1}{4} \frac{\gamma_1 - \gamma_2 + 2 \gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{2}{2} t} - \frac{1}{4} \frac{\gamma_1 - \gamma_2 + 2 \gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{2}{2} t} - \frac{1}{2} e^{-\frac{1}{2} + \frac{\gamma_2 + \gamma_3}{4} t} \cos 2gt. \quad (4.53)
\]

The behavior predicted by this model depends strongly on the decay coefficients \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) that control the rate at which the system loses energy. An interesting but idealized situation arises when one sets \( \gamma_1 = \gamma_2 = 0 \) still keeping \( \gamma_3 \neq 0 \). In this case the only process related to the dissipation is the transition between the dressed states, while there is no possibility to emit a photon to the surrounding environment. Under this condition, equation (4.48) reduces to...
and the corresponding probability for the atom to be in $|g\rangle$ becomes
\begin{equation}
p_g(t) = p_{g,1}(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{\gamma}{2} t} \cos 2g t,
\end{equation}
with $\lim_{t \to \infty} p_g(t) = 1/2$. Although this type of asymptotic behavior occurs also for the damped sinusoids fitting method (see section 4.1), here it is a consequence of the master-equation theoretical analysis. Asymptotic behavior of the atomic ground-state population is due to the fact that the atomic energy is not well defined in the dressed states, to which the evolution is restricted (the system is in a sense trapped between the dressed states, $p_{g,0}(t) = 0$). It is worth of noting, that the damping towards $1/2$ with rate $\gamma_3/4$, predicted in this model, has no counterparts in both phenomenological and microscopic models. Furthermore, from a purely formal viewpoint, large values of $\gamma_3$ (still with $\gamma_1 = \gamma_2 = 0$) causes nearly instantaneous transition of the system from the initial pure state $|e,0\rangle$ to maximally entangled dressed state $|\Omega_{-}\rangle$, which practically kills the oscillations.

Non-zero temperature of the environment complicates the model. Master equation (4.48) acquires three additional terms associated with the possibility of upwards transitions,
\begin{equation}
\dot{\rho} = -i[\Omega, \rho] + \gamma_3 \left( \frac{1}{2} |\Omega_+\rangle \langle \Omega_+| \rho |\Omega_+\rangle \langle \Omega_+| - \frac{1}{4} [\Omega_+ \langle \Omega_+|, \rho]_+ \right) + \gamma_1 \left( \frac{1}{2} |\Omega_0\rangle \langle \Omega_0| \rho |\Omega_+\rangle \langle \Omega_+| - \frac{1}{4} [\Omega_0 \langle \Omega_0|, \rho]_+ \right) + \gamma_a \left( \frac{1}{2} |\Omega_-\rangle \langle \Omega_-| \rho |\Omega_0\rangle \langle \Omega_0| - \frac{1}{4} [\Omega_- \langle \Omega_-|, \rho]_+ \right) + \gamma_2 \left( \frac{1}{2} |\Omega_-\rangle \langle \Omega_-| \rho |\Omega_+\rangle \langle \Omega_+| - \frac{1}{4} [\Omega_- \langle \Omega_-|, \rho]_+ \right) + \gamma_3 \left( \frac{1}{2} |\Omega_-\rangle \langle \Omega_-| \rho |\Omega_0\rangle \langle \Omega_0| - \frac{1}{4} [\Omega_- \langle \Omega_-|, \rho]_+ \right) + \gamma_c \left( \frac{1}{2} |\Omega_-\rangle \langle \Omega_-| \rho |\Omega_0\rangle \langle \Omega_0| - \frac{1}{4} [\Omega_- \langle \Omega_-|, \rho]_+ \right).
\end{equation}

The decay parameters are explained in figure 4.12. Parameters $\gamma_3 = \gamma_c = 0$ reconstruct the model of Scala et al. for $T > 0$K. To solve (4.56) we again employ the damping basis method [61]. We begin with the time-independent operator $\mathcal{L}$ defined by the right-hand side of (4.56), and solve its eigenvalue problem. The off-diagonal eigenvectors
\begin{align}
\mathcal{L}(|\Omega_+\rangle \langle \Omega_-|) &= \left( -i(\Omega_+ - \Omega_-) - \frac{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_c}{4} \right) |\Omega_+\rangle \langle \Omega_-|, \\
\mathcal{L}(|\Omega_+\rangle \langle \Omega_0|) &= \left( -i(\Omega_+ - \Omega_0) - \frac{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_c}{4} \right) |\Omega_+\rangle \langle \Omega_0|, \\
\mathcal{L}(|\Omega_-\rangle \langle \Omega_0|) &= \left( -i(\Omega_- - \Omega_0) - \frac{\gamma_2 + \gamma_a + \gamma_b + \gamma_c}{4} \right) |\Omega_-\rangle \langle \Omega_0|,
\end{align}
can be supplemented by their Hermitian conjugates corresponding to complex-conjugated eigenvalues. The next three operators can be found from the following eigenequation

\[ \mathcal{L}(x|\Omega_+\rangle\langle\Omega_+|+y|\Omega_-\rangle\langle\Omega_-|+z|\Omega_0\rangle\langle\Omega_0|) = \Lambda(x|\Omega_+\rangle\langle\Omega_+|+y|\Omega_-\rangle\langle\Omega_-|+z|\Omega_0\rangle\langle\Omega_0|), \]  

(4.58)
equivalent to the set of equations

\[ \frac{1}{2}(y\gamma_c - x(\gamma_1 + \gamma_3) + z\gamma_a) = \Lambda x, \]  

(4.59a)
\[ \frac{1}{2}(x\gamma_3 - y(\gamma_2 + \gamma_c) + z\gamma_b) = \Lambda y, \]  

(4.59b)
\[ \frac{1}{2}(x\gamma_1 + y\gamma_2 - z(\gamma_a + \gamma_b)) = \Lambda z. \]  

(4.59c)

Their solutions are

\[ x_1 = \gamma_b\gamma_c + \gamma_a(\gamma_2 + \gamma_c), \]  

(4.60a)
\[ y_1 = \gamma_3\gamma_a + \gamma_b(\gamma_1 + \gamma_3), \]  

(4.60b)
\[ z_1 = \gamma_2\gamma_3 + \gamma_1(\gamma_2 + \gamma_c), \]  

(4.60c)
\[ \Lambda_1 = 0, \]  

(4.60d)
\[ x_2 = (\gamma_1 - \gamma_3)(\gamma_c - \gamma_a) - (\gamma_1 + \gamma_3)(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_6 + S), \]  

(4.60e)
\[ y_2 = (\gamma_1 + \gamma_3)(\gamma_3 - \gamma_b) + \gamma_3(\gamma_2 - \gamma_a + \gamma_c + S) - \gamma_1\gamma_b, \]  

(4.60f)
\[ z_2 = -2\gamma_2\gamma_3 + \gamma_1(\gamma_1 - \gamma_2 + \gamma_3 + \gamma_a + \gamma_b - \gamma_c + S), \]  

(4.60g)
\[ \Lambda_2 = -\frac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_a + \gamma_b + \gamma_c + S), \]  

(4.60h)
\[ x_3 = (\gamma_1 - \gamma_3)(\gamma_c - \gamma_a) - (\gamma_1 + \gamma_3)(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_6 - S), \]  

(4.60i)
\[ y_3 = (\gamma_1 + \gamma_3)(\gamma_3 - \gamma_b) + \gamma_3(\gamma_2 - \gamma_a + \gamma_c - S) - \gamma_1\gamma_b, \]  

(4.60j)
\[ z_3 = -2\gamma_2\gamma_3 + \gamma_1(\gamma_1 - \gamma_2 + \gamma_3 + \gamma_a + \gamma_b - \gamma_c - S), \]  

(4.60k)
\[ \Lambda_3 = -\frac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_a + \gamma_b + \gamma_c - S), \]  

(4.60l)
where we have defined

\[ S = \sqrt{(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_a - \gamma_b + \gamma_c)^2 + 4(\gamma_1 - \gamma_2)(\gamma_a - \gamma_c)}. \]  

(4.61)

For the purpose of further reference, it is convenient to have the list of all eigenoperators of \( \mathcal{L} \) implied by (4.56). These are
\[\begin{align*}
\rho_1 &= x_1|\Omega_+\rangle\langle\Omega_+| + y_1|\Omega_-\rangle\langle\Omega_-| + z_1|\Omega_0\rangle\langle\Omega_0|, \\
\rho_2 &= x_2|\Omega_+\rangle\langle\Omega_+| + y_2|\Omega_-\rangle\langle\Omega_-| + z_2|\Omega_0\rangle\langle\Omega_0|, \\
\rho_3 &= x_3|\Omega_+\rangle\langle\Omega_+| + y_3|\Omega_-\rangle\langle\Omega_-| + z_3|\Omega_0\rangle\langle\Omega_0|, \\
\rho_4 &= |\Omega_+\rangle\langle\Omega_-|, \\
\rho_5 &= |\Omega_+\rangle\langle\Omega_0|, \\
\rho_6 &= |\Omega_-\rangle\langle\Omega_0|, \\
\rho_7 &= |\Omega_-\rangle\langle\Omega_+|, \\
\rho_8 &= |\Omega_0\rangle\langle\Omega_+|, \\
\rho_9 &= |\Omega_0\rangle\langle\Omega_-|. \\
\end{align*}\] (4.62a) - (4.62i)

Up to this point all calculations have been carried out without any assumption about the values of \(\gamma_a, \gamma_b\) and \(\gamma_c\). This strategy, however, will no longer pay as the resulting equations would be obscure and difficult to analyze. Since at \(T = 0.8\)K

\[\begin{align*}
\gamma_a &= e^{-\frac{\hbar(c_0 + g)}{3}} \gamma_1 \approx 0.0466327 \gamma_1, \tag{4.63a} \\
\gamma_b &= e^{-\frac{\hbar(c_0 - g)}{3}} \gamma_2 \approx 0.0466328 \gamma_2, \tag{4.63b} \\
\gamma_c &= e^{-\frac{3\hbar}{37}} \gamma_3 \approx 0.999997 \gamma_3, \tag{4.63c}
\end{align*}\]

it is justified to put \(\gamma_a = \epsilon \gamma_1, \gamma_b = \epsilon \gamma_2\) with \(\epsilon = 0.0466\), and \(\gamma_c = \gamma_3\). Figure 4.13 shows that \(\gamma_c \approx \gamma_3\) is valid for a wide range of temperatures. For temperatures of the order of \(10^{-3}\)K the departures from exact equality amount to \(0.003\gamma_3\).

Taking this fact into account the initial state of the system, \(\rho(0) = |e, 0\rangle\langle e, 0|\), expressed in the basis of operators (4.62), can be written as follows

\[\rho(0) = \frac{1}{\mathcal{I}} \sum_{i \in \mathcal{I}} A_i \rho_i, \quad \mathcal{I} = \{1, 2, 3, 4, 7\}, \tag{4.65}\]

with \(A_i\), following from (4.64).

With the knowledge of \(\rho(0)\) it is straightforward to derive \(\rho(t)\) at any time \(t\),
Figure 4.13: The dependence of the factor $e^{-\frac{2g}{\pi T}}$ multiplying $\gamma_3$ in equation (4.63c) on temperature. For a wide range of temperatures we can simply set $\gamma_c = \gamma_3$; $g = 47\pi 10^3$Hz.

\[
\rho(t) = \sum_{i=1}^{3} A_i e^{A_i t} \rho_i + A_4 e^{A_4(g)t} \rho_4 + A_7 e^{A_7(g)t} \rho_7 , \tag{4.66}
\]

where the explicit dependence on coupling $g$ has been indicated.

For $t \to \infty$ the system arrives at a finite-dimensional thermal equilibrium

\[
\lim_{t \to \infty} \rho(t) = \frac{\epsilon}{2\epsilon + 1} |\Omega_+\rangle \langle \Omega_+| + \frac{\epsilon}{2\epsilon + 1} |\Omega_-\rangle \langle \Omega_-| + \frac{1}{2\epsilon + 1} |\Omega_0\rangle \langle \Omega_0|. \tag{4.67}
\]

The coefficients multiplying $|\Omega_\pm\rangle \langle \Omega_\pm|$ are identical as a consequence of our simplifying assumption $e^{-\frac{\hbar(|\omega_0+g|)}{2kT}} \approx e^{-\frac{\hbar(|\omega_0-g|)}{2kT}} \approx \epsilon$. The fact that Davies operators are defined in the dressed-state basis results in $\rho(\infty)$ that commutes with the system Hamiltonian, as it should. Although we do not have an exact formula for $T > 0$K in the phenomenological model, it seems that the asymptotic state will there commute with the free-field Hamiltonian, and thus not with the total one.

The asymptotic equilibrium is no longer a Gibbs state, a fact that follows from our truncated-space model. Finite dimensional thermal equilibria are known to occur in $q$-deformed statistics [47], so probably what we obtain is a state belonging to a $q$-exponential family [46, 62], but the problem requires more detailed studies and is left open here.

The explicit form of the probability we are interested in is, finally,

\[
p_g(t) = \frac{1 + \epsilon}{1 + 2\epsilon} + \frac{2\gamma_3 - \epsilon(\gamma_1 + \gamma_2) - S}{4S(2\epsilon + 1)} e^{-\frac{\gamma_1 + \gamma_2 + 2\gamma_3 + (\gamma_1 + \gamma_2) + S}{4}},
- \frac{2\gamma_3 - \epsilon(\gamma_1 + \gamma_2) + S}{4S(2\epsilon + 1)} e^{-\frac{\gamma_1 + \gamma_2 + 2\gamma_3 + (\gamma_1 + \gamma_2) - S}{4}},
- \frac{1}{2} e^{-\frac{1}{4}(2\gamma_1 + 2\gamma_2)} \cos 2g t, \tag{4.68}
\]
with the asymptotic value

$$\lim_{t \to \infty} p_g(t) = \frac{1 + \epsilon}{1 + 2\epsilon} \approx 0.957,$$  \hfill (4.69)$$

which could be also obtained from equation (4.67).

If we put $T = 0, \gamma_1 = \gamma_2 = 0$, we find the already known fitting formula of Brune et al. [11],

$$p_g(t) = \frac{1}{2} - \frac{1}{2} e^{-\frac{2\pi}{\epsilon} t} \cos 2gt,$$  \hfill (4.70)$$

but with the damping parameter whose interpretation is completely different from what was expected. This simple test suggests that the open-cavity model may be indeed the correct way towards explanation of the discussed experiment. Let us remark that formulas analogous to (4.70) do occur in (semiclassical) Bloch-equation approaches to Rabi oscillations (cf. [49], especially equation (11.4) and figure 11.2).

What yet has to be understood is the meaning of the damping parameters — the next section will be devoted to this problem. Before that, however, let us briefly discuss the modification that arises due to the Gaussian profile.

We assume, as before, that the system is in exact resonance. The assumption has one extremely important technical implication: The eigenvectors $\rho_1, \ldots, \rho_9$, are independent of $g$. Thus, the coupling parameter $g$ enters only into the “off-diagonal” eigenvalues $\Lambda_4, \ldots, \Lambda_9$. This property will allow us to derive an exact formula for $\rho(t)$, even for the Gaussian mode profile. Indeed, repeating the argument presented in the context of the Scala et al. model, we find

$$\rho(t) = A_1 e^{\Lambda_1 t} \rho_1 + A_2 e^{\Lambda_2 t} \rho_2 + A_3 e^{\Lambda_3 t} \rho_3 + A_4 e^{-\frac{2\pi}{\epsilon} \sqrt{\pi} \omega_d t} e^{-2i\epsilon g \sqrt{\pi} \omega_d t} \rho_4 + A_7 e^{-\frac{2\pi}{\epsilon} \sqrt{\pi} \omega_d t} e^{2i\epsilon g \sqrt{\pi} \omega_d t} \rho_7,$$  \hfill (4.71)$$

and

$$p_g(t) = \frac{1 + \epsilon}{1 + 2\epsilon} \left( 1 - \frac{2\gamma_3 - \epsilon(\gamma_1 + \gamma_2) - S}{4S(2\epsilon + 1)} e^{-\frac{2\gamma_3}{4} \frac{2\pi}{\epsilon} \sqrt{\pi} \omega_d t} \right)$$

$$\left( 1 - \frac{-2\gamma_3 - \epsilon(\gamma_1 + \gamma_2) + S}{4S(2\epsilon + 1)} e^{-\frac{2\gamma_3}{4} \frac{2\pi}{\epsilon} \sqrt{\pi} \omega_d t} \right)$$

$$\left( -\frac{1}{2} e^{-\frac{2\pi}{\epsilon} \sqrt{\pi} \omega_d t} \cos 2g \sqrt{\pi} \omega_d t \right).$$  \hfill (4.72)$$

The result again agrees with the Brune et al. recipe, with all the restrictions made on the meaning of real and effective times and parameters. In practice, one also has to include the fact that real measurements involve some uncertainty of $t$ in $\rho(t)$, for example due to collisions of atoms with a background gas and the resulting velocity error. An effective density matrix should therefore be
Figure 4.14: (A) Dependence of $p_g(t)$ (equation (4.68)) on time and $\gamma_3$ based on equation (4.68). The remaining parameters are $g = 47\pi 10^3$ Hz and $\gamma_1 = \gamma_2 = 0.01g$. (B) $p_g(t)$ for a longer time (500 µs) for $\gamma_3 = 0.2g$ (solid), $\gamma_3 = 0.05g$ (dashed) and other parameters as in the left picture. $\gamma_3$ controls the rate at which oscillations decay towards 1/2, whereas $\gamma_1$ and $\gamma_2$ determine the rate at which $p_g(t)$ tends towards 1, reaching it exactly only for $\epsilon = 0$. For $\gamma_1 = \gamma_2 = 0$ oscillations are centered at 1/2.

$$ \rho_{\Delta t}(t) = \int_0^\infty dt' p_{\Delta t}(t, t') \rho(t'), $$ (4.73)

with $p_{\Delta t}(t, t')$ being defined by equation (4.7). Master equation (4.73) satisfies a collisional nondissipative master equation [42, 49, 37] with a double-commutator term $-\frac{\Delta t}{2} [\Omega, [\Omega, \rho_{\Delta t}(t)]]$. The corresponding ground-state probability

$$ p_{g,\Delta t}(t) = \int_0^\infty dt' p_{\Delta t}(t, t') p_g(t'), $$ (4.74)

reconstructs the idealized case by $\lim_{\Delta t \to 0} p_{g,\Delta t}(t) = p_g(t)$. Inserting (4.72) into (4.74) one gets
Figure 4.15: (A) The probability \( p_{g,\Delta t}(t) \) given by (4.75) with \( \gamma_3 = 0 \) and \( \gamma_1 = \gamma_2 = 1772 \text{Hz} \), confronted with the Rabi oscillation data of [11]. Parameters \( \gamma_1 \) and \( \gamma_2 \) are found in the procedure of fitting with exponential energy decay corresponding to \( Q = 7 \cdot 10^7 \), as reported in [11]. (B) An analogous plots but with \( Q = 3.31 \cdot 10^{10} \), \( \gamma_1 = \gamma_2 = 17.73 \text{Hz}, \gamma_3 = 0 \). \( t \) is the true time and \( \Delta t = 2.37 \mu s \), as estimated in [42].

The atomic ground-state population based on the above equation is depicted in figure 4.15.

The disagreement between the experiment and theoretical predictions of the two previous models manifests itself in the lack of physical parameter that could prevent theoretical curves from falling towards 1.

### 4.8 Cavity Q factor versus damping parameters

For a constant \( g \) the average energy confined in the cavity would be \( \overline{E(t)} = \hbar \text{Tr} \Omega \rho(t) = \hbar \Omega(t) \), where

\[
\begin{align*}
p_{g,\Delta t}(t) &= \frac{1 + \epsilon}{1 + 2\epsilon} + \frac{2\gamma_3 - \epsilon(\gamma_1 + \gamma_2)}{4S(2\epsilon + 1)} \left( 1 + \frac{\gamma_1 + \gamma_2 + 2\gamma_3 + \epsilon(\gamma_1 + \gamma_2) + S}{4} \right)^{-\frac{1}{2S}} \\
&\quad - \frac{2\gamma_3 - \epsilon(\gamma_1 + \gamma_2)}{4S(2\epsilon + 1)} \left( 1 + \frac{\gamma_1 + \gamma_2 + 2\gamma_3 + \epsilon(\gamma_1 + \gamma_2) - S}{4} \right)^{-\frac{1}{2S}} \\
&\quad - \frac{1}{2} \left[ \frac{1 + \gamma_1 + \gamma_2 + 2\gamma_3}{4} \Delta t \right]^2 + \frac{4\pi g^2 w^2}{d^2} \Delta t^2 \right]^{-\frac{1}{2S}} \\
&\quad \times \cos \left[ \frac{t}{\Delta t} \arctan \left( 2g \sqrt{\frac{w}{d}} \frac{\Delta t}{1 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3} \right) \right].
\end{align*}
\]
\[
\Omega(t) = \sum_{i=1}^{3} A_i e^{\Lambda_i t} \text{Tr} \Omega \rho_i
\]

\[
= \frac{\omega_0}{2} e^{-1 \frac{2\epsilon}{2\epsilon + 1}} + \frac{\gamma_1 (\omega_0 + 2g) + \gamma_2 (\omega_0 - 2g) + g(\gamma_1 - \gamma_2) - 2\omega_0 \gamma_3 + S \omega_0 e^{-\frac{\gamma_1 + \gamma_2 + 2\gamma_1 + (\gamma_1 + \gamma_2) + \omega_0}{2(\epsilon + 1) t}} - \gamma_1 e(\omega_0 + 2g) + \gamma_2 e(\omega_0 - 2g) + g(\gamma_1 - \gamma_2) - 2\omega_0 \gamma_3 - S \omega_0 e^{-\frac{\gamma_1 + \gamma_2 + 2\gamma_1 + (\gamma_1 + \gamma_2) - \omega_0}{2(\epsilon + 1) t}}}{2S(\epsilon + 1)}.
\]

(4.76)

Asymptotically,

\[
\lim_{t \to \infty} \left( \frac{\Omega(t) + \omega_0}{2} \right) = \omega_0 \left( 1 - \frac{1}{2\epsilon + 1} \right),
\]

(4.77)

where the shift by \(\omega_0/2\) is in order to normalize system’s ground energy to 0 (recall that \(\Omega_0 = -\omega_0/2\)).

It should be kept in mind that \(\epsilon = 0\) does not recover the \(T = 0\)K case, since (4.76) has been derived under the simplifying assumption that \(\gamma_3 = \gamma_c\), so that despite transitions \(|\Omega_0\rangle \to |\Omega_\pm\rangle\) are not allowed, the atom-field system can still circulate between \(|\Omega_+\rangle\) and \(|\Omega_-\rangle\) with the rate determined by \(\gamma_3\).

An interesting case arises if we assume that \(\gamma_1 = \gamma_2 = \gamma\), when \(\Omega(t)\) becomes independent of \(\gamma_3\), and reduces to

\[
\Omega(t) + \frac{\omega_0}{2} = \frac{2\epsilon + e^{-\frac{\gamma_2 + 1}{2} t}}{2\epsilon + 1} \omega_0.
\]

(4.78)

This result can be interpreted as follows. Since \(\gamma_1 = \gamma_2\), both \(|\Omega_+\rangle\) and \(|\Omega_-\rangle\) are indistinguishable from the point of view of the dissipative processes. In other words, transitions between \(|\Omega_-\rangle\) and \(|\Omega_+\rangle\) do not influence the leaks of energy from the system. Moreover, at \(T = 0\)K the decay simplifies even further

\[
\Omega(t) + \frac{\omega_0}{2} = e^{-\frac{\gamma}{2} t} \omega_0.
\]

(4.79)

Coming back to (4.76), with no assumptions about \(\gamma_1\), \(\gamma_2\), \(\gamma_3\) and \(\epsilon\), we can compare it to a simple exponential energy decay,

\[
\Omega(t) - \Omega(\infty) \simeq \left( \Omega(0) - \Omega(\infty) \right) e^{-\frac{\omega_0}{2\epsilon + 1} t} = \frac{\omega_0}{2\epsilon + 1} e^{-\frac{\omega_0}{2\epsilon + 1} t},
\]

(4.80)

where \(Q\) is the cavity quality factor.

To find the optimal set of parameters \(\gamma_1\), \(\gamma_2\) and \(\gamma_3\) we can use the iterative Levenberg-Marquardt algorithm for nonlinear optimization, setting \(\epsilon = 0.0466\) (\(T = 0.8\)K) and \(Q = 7 \cdot 10^7\) as reported in [11]. Straightforward numerical analysis shows that the best fit to the
Figure 4.16: (A) The average $\bar{\Omega}(t) - \bar{\Omega}(\infty)$ of the atom-field system predicted by our model (solid) compared with the exponential decay (dots) modeled by $(\bar{\Omega}(0) - \bar{\Omega}(\infty))e^{-Qt}$, with $Q = 7 \cdot 10^7$ as reported in [11], and $\gamma_1 = \gamma_2 = 1772$Hz found by optimization. (B) The probability $p_g(t)$ predicted according to (4.72) with $\gamma_1 = \gamma_2 = 1772$Hz, $\gamma_3 = 0.07g$ found on the basis of the reported $Q$ factor, confronted with the Rabi oscillation data of [11]. Parameters $\gamma_1$ and $\gamma_2$ are found in the procedure of fitting with a simple exponential energy decay, while $\gamma_3$ minimizes the sum of squared errors. (C) and (D) show analogous plots, but with $Q = 3.31 \cdot 10^{10}$, $\gamma_1 = \gamma_2 = 17.73$Hz, $\gamma_3 = 0.07g$; $t$ is the true time. Various combinations of $\gamma_3$ and $\Delta t$ lead to practically identical plots for $p_g(t, \Delta t)$ (e.g. $\Delta t = 0.5\mu s$, $\gamma_3 = 0.057g$).
right side of (4.80) is obtained for \( \gamma_1 = \gamma_2 = 1772 \text{Hz} \) and arbitrary \( \gamma_3 \), which is shown in Fig. 4.16A. But when inserting these parameters into \( p_g(t) \) we obtain poor agreement with the data (Fig. 4.16B; the best agreement was found for \( \gamma_3 = 0.057 \)).

If, on the other hand, we try to fit the Rabi oscillation data (Fig. 4.16D), the optimal parameters \( \gamma_1 = \gamma_2 = 17.73 \text{Hz}, \gamma_3 = 0.07g \) imply \( Q = 3.31 \cdot 10^{10} \) (Fig. 4.16C). The plots in Fig. 4.16A and Fig. 4.16C will not change if we replace in (4.76) \( g = 47\pi 10^3 \text{Hz} \) by \( g = 0 \). The fits (4.80) are thus insensitive to the exact form of \( g(t) \) within the range of parameters typical of the experiment [11]. Moreover, there is no visible change in the energy decay plots if we replace \( \Omega(t) \) by

\[
\Omega_{\Delta t}(t) = \int_0^\infty dt' p_{\Delta t}(t, t')\Omega(t'),
\]

(4.81)

with \( \Delta t \) even as large as \( 5\mu s \), which is greater than the time sampling step employed in the experiment. In consequence, the discrepancy between \( Q \) given above and the parameter reported in the experiment cannot be explained by finite \( \Delta t \).

Let us now compare the data from [11] with (4.75) under the assumption that \( \gamma_3 = 0 \), that is if the long-wave transition \( |\Omega_+\rangle \rightarrow |\Omega_-\rangle \) is absent. The estimates made in [42] led, for the experiment of Brune et al., to the error \( \Delta t \approx 0.5\mu s \) of the effective time. For the true time the corresponding value is \( \Delta t \approx 2.37\mu s \). Inserting the latter value into (4.75) and assuming \( \gamma_3 = 0 \) we obtain damped Rabi oscillations shown in Fig. 4.15. The agreement is again good.

The conclusion is that the effects described in [42] may be of the same order as those arising from opening the cavity. Since the cavity was in fact open, it seems that the realistic value of \( \Delta t \) was probably smaller than the one discussed in [42].

### 4.9 Dynamics of entanglement between atomic and photon degrees of freedom

Given the generalized open-cavity model of dissipation it is interesting to ask for how long the atoms and photons in the cavity can be regarded as an entangled bipartite system [63, 64]. The problem is important for cavity QED implementations of quantum information processing, and is not entirely obvious since we know that there exist two decoherence time scales. Energy losses are characterized by \( \gamma_1 \) and \( \gamma_2 \) (damping of \( p_g(t) \) towards 1), while \( \gamma_3 \) is responsible for the damping of \( p_g(t) \) towards \( 1/2 \).

A very strong tool for detection of entanglement in low-dimensional systems is the positive partial transpose (PPT) or Peres-Horodecki criterion [65, 66, 67]. According to it, the fact that a given density matrix \( \rho \) of a bipartite system is not PPT implies that the system in question is entangled.

Supplementing the three basis bare states of our atom-field system, \( |10\rangle = |e, 0\rangle, |01\rangle = |g, 1\rangle, |00\rangle = |g, 0\rangle \), by the fourth state \( |11\rangle = |e, 1\rangle \) we obtain a bipartite \( 2 \times 2 \) composite system whose separability is completely characterized by the above partial transposition criterion. Density matrix (4.52) expressed in this bases takes the following form
\[
\rho(t) = \begin{pmatrix}
\rho_{11,11} & \rho_{11,10} & \rho_{10,11} & \rho_{10,10} \\
\rho_{11,01} & \rho_{11,00} & \rho_{10,01} & \rho_{10,00} \\
\rho_{01,11} & \rho_{01,10} & \rho_{00,11} & \rho_{00,10} \\
\rho_{01,01} & \rho_{01,00} & \rho_{00,01} & \rho_{00,00}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \rho_{11,00} & \rho_{10,01} & 0 \\
0 & \rho_{01,10} & \rho_{00,11} & 0 \\
0 & 0 & \rho_{00,00} & \rho_{00,00}
\end{pmatrix}. \tag{4.82}
\]

Its partial transpose with respect to the atomic subsystem

\[
\rho^{T_A}(t) = \begin{pmatrix}
0 & 0 & 0 & \rho_{11,00} \\
0 & \rho_{11,00} & 0 & 0 \\
0 & 0 & \rho_{00,11} & 0 \\
\rho_{10,01} & 0 & 0 & \rho_{00,00}
\end{pmatrix}, \tag{4.83}
\]

has four eigenvalues equal to

\[
\lambda_1 = \langle e, 0 | \rho(t) | e, 0 \rangle \geq 0, \tag{4.84}
\]
\[
\lambda_2 = \langle g, 1 | \rho(t) | g, 1 \rangle \geq 0, \tag{4.85}
\]
\[
\lambda_3 = \frac{1}{2} \left( \langle g, 0 | \rho(t) | g, 0 \rangle + \sqrt{\langle g, 0 | \rho(t) | g, 0 \rangle^2 + 4 \langle e, 0 | \rho(t) | g, 1 \rangle^2} \right) \geq 0, \tag{4.86}
\]
\[
\lambda_4 = \frac{1}{2} \left( \langle g, 0 | \rho(t) | g, 0 \rangle - \sqrt{\langle g, 0 | \rho(t) | g, 0 \rangle^2 + 4 \langle e, 0 | \rho(t) | g, 1 \rangle^2} \right) \leq 0. \tag{4.87}
\]

The last equation implies that the effective measure of entanglement in the system is

\[
\langle e, 0 | \rho(t) | g, 1 \rangle = \frac{1}{4} \frac{\gamma_1 - \gamma_2 + 2\gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{\gamma_1 + 2\gamma_2}{2} t} - \frac{1}{4} \frac{\gamma_1 - \gamma_2 + 2\gamma_3}{\gamma_1 - \gamma_2 + \gamma_3} e^{-\frac{\gamma_1 + \gamma_2 + \gamma_3}{2} t} \sin 2gt. \tag{4.88}
\]

This quantity vanishes for \( t \to \infty \) and, independently, for \( \gamma_1 - \gamma_2 + 2\gamma_3 = 0 \) and \( \sin 2gt = 0 \). In both cases the atom-field system becomes separable.

Figure 4.17 shows dynamics of entanglement of the system described by density matrix (4.71).

**4.10 Conclusions**

The formalism where Davies jump operators are defined in the dressed-state basis is mathematically simpler from the standard one where the jumps are defined by photonic creation and annihilation operators. Simultaneously, when extended to include the long-wave transitions typical of open cavities, the model predicts corrections to Rabi oscillations of the same order as those resulting from uncertainties in measurement of \( t \). The so far ignored fluctuations \(|\Omega_+\rangle \leftrightarrow |\Omega_-\rangle\) turn out to have fundamental implications for coherence properties of the atom–photon system in cavity QED. In particular, these are the transitions that crucially
influence separability properties of states in the cavity and, hence, the attempts of cavity QED implementations of quantum gates have to minimize both $\Delta t$ and $\gamma_3$. Improvements of cavity $Q$ factors as well as lowering the cavity temperature yet lower than 0.8K may be less essential. The fact that the value of $Q$ we have determined on the basis of the data is some 500 times larger from the value reported in the experiment requires further studies. One of the possible interpretations is that the rate at which energy is lost in the cavity depends on $a^\dagger a$, and thus for small light intensities the value of $Q$ may be much larger from the value one would have found for larger numbers of photons. This interpretation is supported by the observation that the missing Davies operator can be regarded as a consequence of $a^\dagger a \otimes B$ system–reservoir interaction, where $B$ is some operator acting in the reservoir Hilbert space.

We are now in position to apply the above results to the problem of cavity QED tests of different field quantization paradigms, in particular to the issue of reducible representations of the algebra of quantum fields. There are reasons to believe that fields quantized in such representations are much less singular than the standard operator-valued distributions, so the question is directly related to various very fundamental issues that have not been subject to quantum optical tests as yet. At the level of vacuum Rabi oscillations the reducibility of representations should be manifested in terms of collapses and revivals in exact vacuum [5], but observability of the effect — if it really exists — will crucially depend on controllability of the coherence losses plaguing cavity QED. These and related issues are discussed in the next chapter.
Chapter 5

Jaynes-Cummings model in lossy cavities. Reducible representations approach

(...)for if experiment should show that it contains just a single ‘element of truth’ that is not in QED, then the alternative theory will have served its purpose.

E. T. Jaynes

The conclusion of the preceding chapter was that an appropriate description of interaction between atoms and a field in open cavities should take into account long-wave transitions between dressed energy levels of the combined system. The corresponding equation can be obtained heuristically starting from the microscopic equation, as well as derived from the very basics of quantum theory of open systems. This chapter goes further and examines the dissipative Jaynes-Cummings model from the reducible representation viewpoint. Particularly interesting is the question whether the unusual phenomena predicted by the theory proposed in this thesis may survive destroying interaction of the system with its environment.

5.1 Bare and dressed states

Let $\mathcal{H}_A$ and $\mathcal{H}_F$ denote Hilbert spaces of an atom and a gas composed of $N$ indefinite-frequency oscillators, respectively. The total Hilbert space of the combined atom-field system can then be written as the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_F$. In order to make calculations feasible and to get reasonable formulas, it is advantageous first to decompose $\mathcal{H}$ into orthogonal subspaces which are invariant under the action of the system Hamiltonian,

\[
H = \hbar \Omega = \hbar \left( \omega R_3 + \omega N_{\omega} + gR_+a_{\omega} + gR_-a_{\omega}^\dagger \right),
\]

(5.1)
where $\omega$ denotes the mode through which subsystems interact (we implicitly assume strict resonance, $\Delta = 0$). Equation (5.1) implies that frequencies of individual oscillators remain unchanged during the evolution, that is

$$H|\phi_{\omega_1 \ldots \omega_N}\rangle = |\phi'_{\omega_1 \ldots \omega_N}\rangle,$$

for any $|\phi_{\omega_1 \ldots \omega_N}\rangle, |\phi'_{\omega_1 \ldots \omega_N}\rangle \in \mathcal{H}$. This leads to the straightforward conclusion that $\mathcal{H}$ can be effectively decomposed into a direct sum of orthogonal subspaces,

$$\mathcal{H} = \bigoplus_{\omega_1, \ldots, \omega_N} \mathcal{H}_A \otimes \mathcal{H}_{\omega_1 \ldots \omega_N},$$

where $\mathcal{H}_{\omega_1 \ldots \omega_N}$ is spanned by vectors $|n_{\omega_1} \ldots n_{\omega_N}\rangle$, invariant under the action of $H$. Therefore, dynamics of the system decomposes into dynamics in orthogonal subspaces $\mathcal{H}_A \otimes \mathcal{H}_{\omega_1 \ldots \omega_N}$ labeled by sequences of frequencies $\omega_1 \ldots \omega_N$. An important parameter associated with these sequences is the number $s$ of occurrences of resonant frequency $\omega$ at which subsystems can interact. For any basis state $|n_{\omega_1} \ldots n_{\omega_N}\rangle \in \mathcal{H}_{\omega_1 \ldots \omega_N}$, this number can be retrieved by applying the frequency of success operator $I_{\omega}$ discussed in chapter 2,

$$I_{\omega}|n_{\omega_1} \ldots n_{\omega_N}\rangle = s \frac{N}{N}|n_{\omega_1} \ldots n_{\omega_N}\rangle.$$

Notice at this point that, although frequencies involved in indexing sequences $\omega_1 \ldots \omega_N$ are not explicitly shown, it is implicitly assumed that there are exactly $s$ occurrences of $\omega$.

The common assumption made in the preceding chapters is that the atom-field system starts from the pure state $|e, 0\rangle \langle e, 0|$. Its reducible analog is

$$\rho(0) = |e, O\rangle \langle e, O| = |e\rangle \langle e| \otimes \sum_{\omega_1, \ldots, \omega_N, \omega'_1, \ldots, \omega'_N} O_{\omega_1} \ldots O_{\omega_N} O'_{\omega'_1} \ldots O'_{\omega'_N} |0_{\omega_1} \ldots 0_{\omega_N}\rangle \langle 0_{\omega'_1} \ldots 0_{\omega'_N}|,$$

provided the vacuum state is identified with $|O\rangle$ defined by equation (2.69). A vacuum is, however, not unique within the reducible representations formalism, so, in principle, also other vacuum states could be considered.

Taking (5.5) as the initial state and making an additional assumption of the environment at temperature $T = 0K$, the underlying representation space $\mathcal{H}_A \otimes \mathcal{H}_{\omega_1 \ldots \omega_N}$ can be effectively restricted to its subspace spanned by the following $s + 2$ orthonormal vectors,

$$|e, 0_{\omega_1} \ldots 0_{\omega_N}\rangle = |+\rangle|0_{\omega_1} \ldots 0_{\omega \ldots 0_{\omega_N}}\rangle = |1\rangle,$$

$$|g, 1^{(1)}_{\omega_1 \ldots \omega_N}\rangle = |-\rangle|0_{\omega_1} \ldots 1_{\omega} \ldots 0_{\omega_N}\rangle = |2\rangle,$$

$$\vdots$$

$$|g, 1^{(s)}_{\omega_1 \ldots \omega_N}\rangle = |-\rangle|0_{\omega_1} \ldots 0_{\omega} \ldots 1_{\omega} \ldots 0_{\omega_N}\rangle = |s + 1\rangle,$$

$$|g, 0_{\omega_1} \ldots 0_{\omega_N}\rangle = |-\rangle|0_{\omega_1} \ldots 0_{\omega} \ldots 0_{\omega} \ldots 0_{\omega_N}\rangle = |s + 2\rangle.$$
This corresponds to the restriction of the total Hilbert space to that spanned by \( |e, 0 \rangle \), \( |g, 1 \rangle \) and \( |g, 0 \rangle \) discussed in the context of irreducible representations. Clearly, \( |g, 1^{(i)}_{\omega_1 \ldots \omega_N} \rangle \), \( i \in \{1, \ldots, s\} \), denotes the state of the atom-field system with the atom in the lower state \( |g \rangle \), and \( i \)-th oscillator at resonant frequency \( \omega \) singly excited. States (5.6) can be identified as the bare states of the Hamiltonian (5.1). The dimension of the subspace is intimately related to the value of parameter \( s \). For \( s = 0 \) the dimension is 2, while for \( s = N \) it assumes maximal value equal to \( N + 2 \). It is good to realize that \( s = 0 \) corresponds to infinitely many subspaces \( \mathcal{H}_{\omega_1 \ldots \omega_N} \) such that \( \omega' \neq \omega \). In all such subspaces the corresponding Hamiltonian, \( \hbar \Omega_{\omega_1 \ldots \omega_N} \), reduces to the operator \( \omega R_3 \) and thus the dynamics of the system is trivial. \( s = N \), on the other hand, corresponds to a single subspace where all field-oscillators are at resonance with the atomic frequency \( \omega \).

Denote by \( \Pi_{\omega_1 \ldots \omega_N} \) the projector onto the subspace \( \mathcal{H}_{\omega_1 \ldots \omega_N} \) of \( \mathcal{H} \), and by \( \pi_{\omega_1 \ldots \omega_N} \) the projector on the \((s + 2)\)-dimensional subspace of \( \mathcal{H}_{\omega_1 \ldots \omega_N} \) spanned by (5.6). The relevant dressed states are then the eigenstates of

\[
\Omega_{\omega_1 \ldots \omega_N} = \Omega \pi_{\omega_1 \ldots \omega_N}.
\]

It is instructive to write operator (5.7) in the basis of bare states (5.6). We have

\[
\Omega(s) := \Omega_{\omega_1 \ldots \omega_N} = \begin{pmatrix}
\frac{\omega}{2} & \frac{g}{\sqrt{N}} & \frac{g}{\sqrt{N}} & \cdots & \frac{g}{\sqrt{N}} & 0 \\
\frac{g}{\sqrt{N}} & \frac{\omega}{2} & 0 & \cdots & 0 & 0 \\
\frac{g}{\sqrt{N}} & 0 & \frac{\omega}{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{g}{\sqrt{N}} & 0 & 0 & \cdots & \frac{\omega}{2} & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{\omega}{2}
\end{pmatrix}.
\]

As we can see, \( \Omega(s) \) does not explicitly depend on the concrete values of frequencies that index a given subspace of \( \mathcal{H}_{\omega_1 \ldots \omega_N} \), but only on the number \( s \) of times the resonant frequency \( \omega \) occurs in the indexing sequence \( \omega_1 \ldots \omega_N \). After some algebra it can be shown that the eigenvalues of \( \Omega(s) \) are

\[
\Omega_{\pm}(s) = \frac{\omega}{2} \pm \frac{g}{\sqrt{N}},
\]

\[
\Omega_1(s) = \Omega_2(s) = \cdots = \Omega_{s-1}(s) = \frac{\omega}{2},
\]

\[
\Omega_0(s) = -\frac{\omega}{2},
\]

with the corresponding orthonormal eigenvectors.
\[ |\Omega_0(s)\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (5.10a) \]

\[ |\Omega_1(s)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, |\Omega_k(s)\rangle = \begin{pmatrix} \frac{1}{\sqrt{k(k+1)}} \\ \frac{1}{\sqrt{k(k+1)}} \\ \vdots \\ \frac{1}{\sqrt{k(k+1)}} \\ -\frac{1}{k+1} \\ \vdots \\ 0 \end{pmatrix}, \ldots, |\Omega_{s-1}(s)\rangle = \begin{pmatrix} \frac{1}{\sqrt{(s-1)s}} \\ \frac{1}{\sqrt{(s-1)s}} \\ \vdots \\ \frac{1}{\sqrt{(s-1)s}} \\ -\sqrt{\frac{s-1}{s}} \\ 0 \end{pmatrix}, \quad (5.10b) \]

and

\[ |\Omega_-(s)\rangle = \frac{1}{\sqrt{2s}} \begin{pmatrix} -\sqrt{s} \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad |\Omega_+(s)\rangle = \frac{1}{\sqrt{2s}} \begin{pmatrix} \sqrt{s} \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}. \quad (5.10c) \]

Notice that energetic structure of the system appears here to be richer than that for irreducible representations; this is due to the \((s-1)\)-fold degenerate eigenvalue \(\omega/2\). It will turn out, however, that these extra energetic levels do not take part in dissipation. In order to avoid possible confusion let us remind that equations (5.10) define an orthonormal basis in the relevant subspace of \(H_A \otimes H_{\omega_1 \cdots \omega_N}\), and not in the full Hilbert space \(\mathcal{H}\).

The relation between the bare states given by (5.6) and the just found dressed states (5.10) is the following
\begin{align}
|\Omega_0(s)\rangle &= |g,0_{\omega_1...\omega_N}\rangle, \quad (5.11a) \\
|\Omega_+(s)\rangle &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{s}} \sum_{i=1}^{s} |g,1_{\omega_1...\omega_N}^{(i)}\rangle + |e,0_{\omega_1...\omega_N}\rangle \right), \quad (5.11b) \\
|\Omega_-(s)\rangle &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{s}} \sum_{i=1}^{s} |g,1_{\omega_1...\omega_N}^{(i)}\rangle - |e,0_{\omega_1...\omega_N}\rangle \right), \quad (5.11c) \\
|\Omega_k(s)\rangle &= \frac{1}{\sqrt{k(k+1)}} \sum_{i=1}^{k} |g,1_{\omega_1...\omega_N}^{(i)}\rangle - \sqrt{\frac{k}{k+1}} |g,1_{\omega_1...\omega_N}^{(k+1)}\rangle. \quad k = 1,\ldots,s-1. \quad (5.11d)
\end{align}

In particular
\begin{equation}
|e,0_{\omega_1...\omega_N}\rangle = \frac{1}{\sqrt{2}} \left( |\Omega_+(s)\rangle - |\Omega_-(s)\rangle \right), \quad (5.12)
\end{equation}

which is much the same as the well-known formula obtained for irreducible representations.

### 5.2 Master equation at zero temperature

The next step is to derive the equation of motion appropriate to describe atoms interacting with the reducible-representations field in open-cavities. This can be done by formally repeating the steps described in Appendix, that finally led to generalized master equation (4.48).

The reducible analog of the system-reservoir Hamiltonian is
\begin{equation}
H_{S+R} = H + H_R + H_{\text{int}}, \quad (5.13)
\end{equation}
where $H$ is given by equation (5.1), the explicit form of $H_R$ is not relevant from the point of view of this discussion, and $H_{\text{int}}$ corresponds to the system-environment interaction,
\begin{equation}
H_{\text{int}} = \hbar \left\{ \alpha (\hat{a}_\omega + \hat{a}_\omega^\dagger) + \beta \hat{N}_{\omega} \right\} \otimes B, \quad \alpha, \beta \in \mathbb{R}. \quad (5.14)
\end{equation}
The free-field operator $\hat{N}_{\omega}$ is used here in analogy to the standard free-field operator $\hat{a}_\omega^\dagger \hat{a}_\omega$. Note that in the standard formulation, the counterpart of the hamiltonian (5.14) with $\beta = 0$ leads to the Scala et al. microscopic model of dissipation which, at $T = 0$K, excludes transitions between the dressed states. These transitions should be, however, included into analyzes of setups involving open cavities.

Transformations induced by (5.14) leave indexing sequences of frequencies of field-oscillators unchanged. Formally, this means that $[H_{S+R}; \Pi_{\omega_1...\omega_N}] = 0$. The full system-reservoir Hilbert space $\mathcal{H}_{S+R}$ can, therefore, be split into a direct sum of subspaces,
\begin{equation}
\mathcal{H}_{S+R} = \bigoplus_{\omega_1,...,\omega_N} \mathcal{H}_A \otimes \mathcal{H}_{\omega_1...\omega_N} \otimes \mathcal{H}_R. \quad (5.15)
\end{equation}
As has been already noted, in particular case of subspaces with \( s = 0 \) the atom and the field are unable to exchange photons. Equation (5.14) implies in addition that these spaces are free of dissipation.

Our primary goal is to compute the evolution of the atomic ground-state population \( p_g(t) = \text{Tr} \rho(t)|g\rangle\langle g| \), where \( \rho(t) \) describes the atom-field system. We have

\[
p_g(t) = \text{Tr} |g\rangle\langle g| \rho(t)
= \sum_{\omega_1,...,\omega_N} \sum_{\omega'_1,...,\omega'_N} \text{Tr} |g\rangle\langle g| \Pi_{\omega_1,...,\omega_N} \rho(t) \Pi_{\omega'_1,...,\omega'_N}
= \sum_{\omega_1,...,\omega_N} \text{Tr} |g\rangle\langle g| \Pi_{\omega_1,...,\omega_N} \rho(t) \Pi_{\omega_1,...,\omega_N}
= \sum_{\omega_1,...,\omega_N} \text{Tr} |g\rangle\langle g| \pi_{\omega_1,...,\omega_N} \rho(t) \pi_{\omega_1,...,\omega_N}.
\]

(5.16)

(5.17)

The transition from equation (5.16) to (5.17) is based on the assumption that at \( T = 0 \) the atom-field system cannot be led out of the subspace spanned by vectors (5.6) or (5.10). Concerning equation (5.17), note that the effective quantity to be found is not the system total density matrix \( \rho(t) \) but rather its projected version

\[
\rho_{\omega_1,...,\omega_N}(t) = \pi_{\omega_1,...,\omega_N} \rho(t) \pi_{\omega_1,...,\omega_N},
\]

(5.18)

whose knowledge allows us to construct the global solution \( \rho(t) \). Following the line of reasoning of [38], the master equation for \( \rho_{\omega_1,...,\omega_N} \) at \( T = 0 \) can be written as

\[
\dot{\rho}_{\omega_1,...,\omega_N}(t) = -i[\Omega_{\omega_1,...,\omega_N}, \rho_{\omega_1,...,\omega_N}(t)]
+ \sum_{\omega > 0} \gamma(\omega) \left( A_{\omega_1,...,\omega_N}(\omega) \rho_{\omega_1,...,\omega_N}(t) A_{\omega_1,...,\omega_N}^\dagger(\omega) - \frac{1}{2} [A_{\omega_1,...,\omega_N}^\dagger(\omega) A_{\omega_1,...,\omega_N}(\omega), \rho_{\omega_1,...,\omega_N}(t)]_+ \right).
\]

(5.19)

\( A_{\omega_1,...,\omega_N}(\omega) \) are jump operators defined as

\[
A_{\omega_1,...,\omega_N}(\omega) = \sum_{\epsilon' - \epsilon = \omega} P_{\omega_1,...,\omega_N}(\epsilon) A_{\omega_1,...,\omega_N}(\epsilon') P_{\omega_1,...,\omega_N}(\epsilon'),
\]

(5.20)

with \( P_{\omega_1,...,\omega_N}(\epsilon) \) being spectral projectors on subspaces associated with the eigenvalue \( \epsilon \) of \( \Omega_{\omega_1,...,\omega_N} \), and

\[
A_{\omega_1,...,\omega_N} = \pi_{\omega_1,...,\omega_N} \left( \alpha(a_\omega + a_\omega^\dagger) + \beta N_\omega \right).
\]

(5.21)

Employing (5.11) and the explicit form of \( A_{\omega_1,...,\omega_N} \), the jump operators can be written as
\[ A_{\omega_1\ldots\omega_N}(\Omega_+(s) - \Omega_0(s)) = \frac{\alpha}{\sqrt{2}} \sqrt{\frac{s}{N}} |\Omega_0(s)\rangle \langle \Omega_+(s)|, \quad (5.22a) \]
\[ A_{\omega_1\ldots\omega_N}(\Omega_+(s) - \Omega_-(s)) = \frac{\beta}{2} |\Omega_-(s)\rangle \langle \Omega_+(s)|, \quad (5.22b) \]
\[ A_{\omega_1\ldots\omega_N}(\Omega_-(s) - \Omega_0(s)) = \frac{\alpha}{\sqrt{2}} \sqrt{\frac{s}{N}} |\Omega_0(s)\rangle \langle \Omega_-(s)|, \quad (5.22c) \]
\[ A_{\omega_1\ldots\omega_N}(\Omega_+(s) - \Omega_k(s)) = A_{\omega_1\ldots\omega_N}(\Omega_k(s) - \Omega_0(s)) = 0, \quad k = 1, \ldots, s - 1. \quad (5.22d) \]

An important and rather unexpected result is that jumps involving \( |\Omega_k(s)\rangle \), \( k = 1, \ldots, s - 1 \), are not allowed. This follows from the explicit form of \( \tilde{N}_\omega \) given by (2.46), which, for \( N > 1 \), should not be identified with \( \tilde{N}_\omega = a_\omega^\dagger a_\omega \). The inclusion of the term \( \tilde{N}_\omega \) into (5.14) would generate also the jumps involving \( |\Omega_k(s)\rangle \). Note, however, that \( [a_\omega, \tilde{N}_\omega] = L_\omega a_\omega \), so the algebra involving \( \tilde{N}_\omega \) is not a Lie algebra.

Finally, inserting (5.22) into (5.19), the master equation for \( \rho_{\omega_1\ldots\omega_N}(t) \) becomes

\[
\dot{\rho}_{\omega_1\ldots\omega_N} = -i[\Omega_{\omega_1\ldots\omega_N}, \rho_{\omega_1\ldots\omega_N}] + \gamma(\Omega_+(s) - \Omega_0(s))\alpha^2 \frac{s}{N} \left( \frac{1}{2} |\Omega_0(s)\rangle \langle \Omega_+(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_+(s)\rangle \langle \Omega_0(s)| - \frac{1}{4} |\Omega_+(s)\rangle \langle \Omega_+(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_+(s)\rangle \langle \Omega_+(s)| \right) + \\
+ \gamma(\Omega_-(s) - \Omega_0(s))\alpha^2 \frac{s}{N} \left( \frac{1}{2} |\Omega_0(s)\rangle \langle \Omega_-(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_-(s)\rangle \langle \Omega_0(s)| - \frac{1}{4} |\Omega_-(s)\rangle \langle \Omega_-(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_-(s)\rangle \langle \Omega_0(s)| \right) + \\
+ \gamma(\Omega_+(s) - \Omega_-(s))\frac{\beta^2}{2} \left( \frac{1}{2} |\Omega_-(s)\rangle \langle \Omega_+(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_+(s)\rangle \langle \Omega_-(s)| - \frac{1}{4} |\Omega_+(s)\rangle \langle \Omega_+(s)| \rho_{\omega_1\ldots\omega_N} |\Omega_+(s)\rangle \langle \Omega_-(s)| \right). \quad (5.23) \]

The parameters \( \gamma_1 = \gamma(\Omega_+(s) - \Omega_0(s))\alpha^2 \), \( \gamma_2 = \gamma(\Omega_-(s) - \Omega_0(s))\alpha^2 \), and \( \gamma_3 = \gamma(\Omega_+(s) - \Omega_-(s))\beta^2/2 \) are related to the system-reservoir interaction Hamiltonian in a way that is identical to that discussed for irreducible representations. In order to have a well defined limit \( N \to \infty \), we assume they are independent of \( s \). However, \( \gamma_1 \) and \( \gamma_2 \) are additionally multiplied by \( s/N \), a fact that introduces an \( s \)-dependence into \( \gamma_1(s) = \gamma_1 s/N \), \( \gamma_2(s) = \gamma_2 s/N \), keeping \( \gamma_3 \) independent of \( s \). Up to this modification, equation (5.23) has a structure identical to that of (4.48).

Before going further notice that \( \rho_{\omega_1\ldots\omega_N} \) is not a density matrix. Indeed,

\[
\text{Tr} \rho_{\omega_1\ldots\omega_N}(t) = \text{Tr} \Pi_{\omega_1\ldots\omega_N} \rho(t) = p_{\omega_1\ldots\omega_N} < 1, \quad (5.24) \]

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where \( p_{\omega_1 \ldots \omega_N} \) is the probability of finding the frequency sequence \( \omega_1 \ldots \omega_N \) if one randomly and independently selects each \( \omega_i \). Alternatively, this can be interpreted as the probability that the system dynamics happens in \( \mathcal{H}_A \otimes \mathcal{H}_{\omega_1 \ldots \omega_N} \). The probability of finding \( \omega_i \) equals \( Z_{\omega_i} = |O_{\omega_i}|^2 \), so that \( p_{\omega_1 \ldots \omega_N} = Z_{\omega_1} \ldots Z_{\omega_N} \). It is more convenient to work with normalized solutions of (5.23), that is

\[
\varrho_{\omega_1 \ldots \omega_N}(t) = \frac{\rho_{\omega_1 \ldots \omega_N}(t)}{\rho_{\omega_1 \ldots \omega_N}(t)} \quad \text{Tr} \varrho_{\omega_1 \ldots \omega_N} = 1. \tag{5.25}
\]

Normalized \( \varrho_{\omega_1 \ldots \omega_N}(t) \) should not be confused with \( \rho_{\omega_1 \ldots \omega_N}(t) \). The former guarantees, for instance, that probabilities of measurements in the respective subspaces are summable to unity.

Based on (5.5), the normalized initial state can now be written as

\[
\varrho_{\omega_1 \ldots \omega_N}(0) = |e, 0_{\omega_1 \ldots \omega_N}\rangle \langle e, 0_{\omega_1 \ldots \omega_N}| = \frac{1}{2} \left( |\Omega_+(s)\rangle \langle \Omega_+(s)| + |\Omega_-(s)\rangle \langle \Omega_-(s)| - |\Omega_+(s)\rangle \langle \Omega_-(s)| - |\Omega_-(s)\rangle \langle \Omega_+(s)| \right). \tag{5.26}
\]

This is the point where the solution (4.52) obtained for irreducible representations can be directly exploited. Making use of it, its reducible analog can be immediately written as

\[
\varrho_{\omega_1 \ldots \omega_N}(t) = \frac{1}{2} \frac{1}{\gamma_1(s) - \gamma_2(s) + \gamma_3} e^{-\gamma_1(s)/2} \left( \begin{array}{c}
(\gamma_1(s) - \gamma_2(s) + \gamma_3)|\Omega_+(s)\rangle \langle \Omega_+(s)| + \gamma_3|\Omega_-(s)\rangle \langle \Omega_-(s)| \\
+ (\gamma_1(s) - \gamma_2(s))|\Omega_0(s)\rangle \langle \Omega_0(s)|
\end{array} \right)
\]

\[
+ \left( \frac{\gamma_1(s) - \gamma_2(s) + 2\gamma_3}{2\gamma_1(s) - \gamma_2(s) + \gamma_3} e^{-\gamma_2(s)/2} \left( |\Omega_-(s)\rangle \langle \Omega_-(s)| - |\Omega_0(s)\rangle \langle \Omega_0(s)| \right) \right)
\]

\[
+ \frac{1}{2} e^{-2g\sqrt{s/N}t} e^{-\gamma_1(s)/4} |\Omega_+(s)\rangle \langle \Omega_+(s)|
\]

\[
- \frac{1}{2} e^{2g\sqrt{s/N}t} e^{-\gamma_1(s)/4} |\Omega_-(s)\rangle \langle \Omega_+(s)|. \tag{5.27}
\]

The conditional probability of finding the atom in its ground state, under the condition that the frequency sequence is \( \omega_1 \ldots \omega_N \) (recall that the structure of this sequence encodes the value of \( s \)), then reads

\[
p_g(s, t) = p_{\varrho_{\omega_1 \ldots \omega_N}}(s, t) = \text{Tr} |g\rangle \langle g| \varrho_{\omega_1 \ldots \omega_N}(t)
\]

\[
= 1 - \frac{\gamma_1(s) - \gamma_2(s) + 2\gamma_3}{4\gamma_1(s) - \gamma_2(s) + \gamma_3} e^{-\gamma_2(s)/2} - \frac{\gamma_1(s) - \gamma_2(s)}{4\gamma_1(s) - \gamma_2(s) + \gamma_3} e^{-\gamma_1(s)/2}
\]

\[
- \frac{1}{2} e^{-\gamma_2(s)/4} \cos 2g\sqrt{s/N}t. \tag{5.28}
\]

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It is instructive to confront this formula with the one obtained in the irreducible formalism with $[\hat{a}, \hat{a}^\dagger] = \mathcal{Z}$, for some constant $\mathcal{Z} > 0$ (if $\mathcal{Z} < 0$, then $\hat{a}$ is a creation operator):

$$
p_g^{\text{irr}}(t) = 1 - \frac{1}{4} \frac{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z} + 2 \gamma_3}{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z} + \gamma_3} e^{-\frac{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z}}{2} t} - \frac{1}{4} \frac{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z} + \gamma_3}{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z} + \gamma_3} e^{-\frac{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z}}{2} t} - \frac{1}{2} e^{-\frac{\gamma_1 \mathcal{Z} - \gamma_2 \mathcal{Z} + \gamma_3}{4} t} \cos 2g\sqrt{\mathcal{Z}} t.
$$

(5.29)

Obviously, $\mathcal{Z}$ occurs in (5.29) in the same place as $s/N$ in (5.28). This is consistent with the fact that in subspaces characterized by $s$, the right-hand side of an analogous commutator involves $s/N$. All irreducible representations imply, therefore, the same physical result provided one defines observable parameters by their renormalized forms: $g_{\text{ph}} = g\sqrt{\mathcal{Z}}$, $\gamma_{1,\text{ph}} = \gamma_1 \mathcal{Z}$, $\gamma_{2,\text{ph}} = \gamma_2 \mathcal{Z}$, $\gamma_{3,\text{ph}} = \gamma_3$.

Returning to the reducible representation, the atomic ground-state probability we are looking for is the weighted sum of the form

$$
p_g(t) = \sum_{\omega_1, \ldots, \omega_N} p_{\omega_1 \ldots \omega_N} \text{Tr} [g] \langle g | g_{\omega_1 \ldots \omega_N} (t) = \sum_{\omega_1, \ldots, \omega_N} p_{\omega_1 \ldots \omega_N} p_g(s, t)
$$

$$
= \sum_{s=0}^{N} \binom{N}{s} Z_{\omega}^s (1 - Z_{\omega})^{N-s} p_g(s, t),
$$

(5.30)

which is consistent with the heuristics proposed in section 3.3.

Equation (5.30) is basically the final formula that should be compared with experiment. Before we do that, however, the bare parameters $\gamma_1$, $\gamma_2$, $\gamma_3$, and $g$, have to be related to their physical, renormalized counterparts. In order to do so, let consider the asymptotic limit $N \to \infty$. The weak law of large numbers implies immediately

$$
\lim_{N \to \infty} p_g(t) = 1 - \frac{1}{4} \frac{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega} + 2 \gamma_3}{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega} + \gamma_3} e^{-\frac{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega}}{2} t} - \frac{1}{4} \frac{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega} + \gamma_3}{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega} + \gamma_3} e^{-\frac{\gamma_1 Z_{\omega} - \gamma_2 Z_{\omega}}{2} t} - \frac{1}{2} e^{-\frac{\gamma_1 Z_{\omega} + \gamma_2 Z_{\omega} + \gamma_3}{4} t} \cos 2g\sqrt{Z_{\omega}} t.
$$

(5.31)

g\sqrt{Z_{\omega}} is the effective physical coupling. Defining $Z = \max_{\omega} \{Z_{\omega}\}$, we note that the formula involves an automatic cutoff $\chi_{\omega} = Z_{\omega}/Z$, $0 \leq \chi_{\omega} \leq 1$. The observable parameters are identified with $g_{\text{ph}} = g\sqrt{Z}$, $\gamma_{1,\text{ph}} = \gamma_1 Z$, $\gamma_{2,\text{ph}} = \gamma_2 Z$, and $\gamma_{3,\text{ph}} = \gamma_3$. This result supports the correspondence principle formulated before: The reducible-representation asymptotic formula reconstructs exactly the standard one under the assumption that $\chi_{\omega} = 1$.

Taking these facts into account, the formula for $p_g(t)$, valid for all values of $1 \leq N < \infty$, finally becomes
The limit \( \lim_{N \to \infty} p_g(t) \) reconstructs (5.29) with \( Z = Z \). Numerical simulations show that the larger \( N \), the less important the exact value of \( Z \). For \( N \) of the order of \( 10^5 \) or higher, plots of \( p_g(t) \) are insensitive to changes of \( Z \) if the product \( N_{ph} = NZ \) is kept constant. Such a limit, \( N \to \infty \) with \( N_{ph} = \text{const} \), is precisely a thermodynamic limit with finite and fixed number of oscillators at mode \( \omega \). Indeed, we have

\[
\lim_{N \to \infty} \frac{s}{N} = \lim_{N \to \infty} \frac{sZ}{N_{ph}} = Z \quad \Longrightarrow \quad \frac{s}{N_{ph}} \to 1.
\]

The thermodynamic limit implicitly removes the cutoff since \( Z = N_{ph}/N \to 0 \) implies shifting of the cutoff to infinity. Requirements of mathematical consistency imply that \( N \) is finite. For physical reasons, however, \( N \) must be very large, and \( Z \) small but non-zero.

Let us now assume that the cavity is identical to the one employed in Brune et al. experiment, where the mode has a Gaussian structure. Denoting by \( d \) and \( w \) the cavity length and the Gaussian width, we get the effective probability

\[
\tilde{p}_g(t) = \sum_{s=0}^{N} \binom{N}{s} Z^s (1 - Z)^{N-s} \times \\
\times \left\{ 1 - \frac{1}{4} \frac{s}{NZ} (\gamma_{1,ph} - \gamma_{2,ph}) + 2 \gamma_3 e^{-\frac{\gamma_{2,ph} s t}{2\sqrt{\pi} w/d}} - \frac{1}{4} \frac{s}{NZ} (\gamma_{1,ph} - \gamma_{2,ph}) + \gamma_3 e^{-\frac{\gamma_{1,ph} s t}{2\sqrt{\pi} w/d}} - \frac{1}{2} e^{-\frac{\gamma_{1,ph} s t}{2\sqrt{\pi} w/d}} \right\} \cos 2g_{ph} \sqrt{\frac{s}{NZ}} t.
\]

where \( t \) denotes here the effective time. Analogously, the Gaussian-mode correction to the irreducible case can be written as

\[
\tilde{p}_g^{irr}(t) = 1 - \frac{1}{4} \gamma_1 Z - 2 \gamma_3 e^{-\frac{\gamma_{2,ph} s t}{2\sqrt{\pi} w/d}} - \frac{1}{4} \gamma_1 Z - \gamma_2 e^{-\frac{\gamma_{1,ph} s t}{2\sqrt{\pi} w/d}} t - \frac{1}{2} e^{-\frac{\gamma_{1,ph} s t}{2\sqrt{\pi} w/d}} \cos 2g \sqrt{Z} t.
\]

Figure 5.1 shows time-dependence of \( \tilde{p}_g(t) \) for \( N = 10^5 \) and various values of \( N_{ph} \). Even more suggestive is the plot of the difference \( |\tilde{p}_g^{irr}(t) - \tilde{p}_g(t)| \), shown in figure 5.2, compared
Figure 5.1: Probability \( \tilde{p}_g(t) \), equation (5.34), of finding the atom in the lower state \(|g\rangle\) for \( N_{ph} = 1 \) (dotted), \( N_{ph} = 100 \) (dashed), and \( N_{ph} = 400 \) (solid), with \( N = 10^5 \) for all the three cases. \( t \) is the effective time. The filled circles represent an analogous probability obtained for irreducible representations. Error bars taken from the experiment of Brune et al. provide a natural measure of distance between predictions of the alternative theories. For higher values of \( N_{ph} \), say \( N_{ph} = 1000 \), the reducible representation becomes indistinguishable from the irreducible one. The remaining parameters are: \( g = 47\pi 10^3 \) Hz, \( \gamma_1 = \gamma_2 = 83.912 \) Hz and \( \gamma_3 = 0.07g \).

with the error bars taken from the data of Brune et al. [11]. It is clear that this concrete experiment cannot discriminate between the limit \( N \to \infty \) (i.e. the standard theory based on irreducible representations) and the alternative non-Wightmanian theory\(^1\) with any finite \( N_{ph} > 400 \).

### 5.3 Energy decay

Identification of physical parameters with the renormalized ones is supported by the analysis of energy losses. In irreducible representations with \([\hat{a}, \hat{a}^\dagger] = Z1\), the average energy of the atom-field system inside of the cavity, at \( T = 0K \), \( E(t) = \hbar \Omega(t)^{irr} = \hbar \text{Tr} \Omega \rho(t) \), is given by

\[
\Omega(t)^{irr} = -\frac{\omega}{2} + \frac{1}{2} \left\{ \frac{\omega}{\gamma_1 Z - \gamma_2 Z + \gamma_3} + g\sqrt{Z} \frac{\gamma_1 Z - \gamma_2 Z + 2\gamma_3}{\gamma_1 Z - \gamma_2 Z + \gamma_3} \right\} e^{-\frac{\gamma_1 Z - \gamma_2 Z + \gamma_3}{2} t} \\
+ \frac{1}{2} \left\{ \frac{\gamma_1 Z - \gamma_2 Z + 2\gamma_3}{\gamma_1 Z - \gamma_2 Z + \gamma_3} \right\} (\omega - g\sqrt{Z}) e^{-\frac{\gamma_1 Z - \gamma_2 Z + \gamma_3}{2} t}.
\]

Equation (5.36)

In the preceding chapter we have shown that best fits to experimental data are found for \( \gamma_1 = \gamma_2 \). Inserting \( \gamma = \gamma_1 = \gamma_2 \) into (5.36), we get

\(^1\)Indeed, the theory is non-Wightmanian since a vacuum is not unique [68, 69].
Figure 5.2: The difference $|\tilde{p}_g^{\text{irr}}(t) - \tilde{p}_g(t)|$, with $\tilde{p}_g^{\text{irr}}(t)$ and $\tilde{p}_g(t)$ given by (5.35) and (5.34), respectively. $t$ is the effective time. $N_{\text{ph}} = 400$ (solid), $N_{\text{ph}} = 1000$ (dashed), $N_{\text{ph}} = 5000$ (dotted), $N = 10^5$. Stars represent the error bars taken from the experiment of Brune et al. The curves remain practically unchanged for higher $N$, so the plots survive the thermodynamic limit with $N_{\text{ph}} = \text{const.}$

$$\Omega(t)^{\text{irr}} = -\omega^2 + \omega e^{-\gamma Z t} + q \sqrt{Z} e^{-\gamma Z t} (e^{-\gamma Z t} - 1).$$

(5.37)

It is evident that the energy damping parameter is $\gamma Z$, and not just $\gamma$.

The reducible-representation result is similar,

$$\text{Tr} \, \Omega_{\omega_1 \ldots \omega_N}(t) = -\frac{\omega^2}{2} + \frac{1}{2} \left\{ \frac{\omega}{\gamma_1(s) - \gamma_2(s)} + \frac{1}{N} \right\} \left[ \frac{s}{N} \gamma_1(s) - \gamma_2(s) + 2 \gamma_3 \right] e^{-\gamma_1(s) + \gamma_3 t}$$

$$+ \frac{1}{2} \gamma_1(s) - \gamma_2(s) + 2 \gamma_3 \left( \omega - \sqrt{\frac{s}{N}} \right) e^{-\frac{\gamma_1(s)}{2} t}.$$  

(5.38)

As before, the right-hand side depends on $s$ and not on the exact form of the sequence $\omega_1 \ldots \omega_N$. Repeating the reasoning from the previous sections, we find

$$\Omega(t) = \sum_{s=0}^{N} \binom{N}{s} Z^s (1 - Z)^{N-s} \Omega(s, t).$$

(5.40)

Figure 5.3 compares the two expressions for various values of $N_{\text{ph}}$, after having renormalized the parameters.
5.4 Vacuum collapses and revivals for $N_{ph} < \infty$

The discussed Rabi-oscillation data do not distinguish between the two alternative forms of field quantization: One can always increase the value of $N_{ph}$ and produce a theory indistinguishable from the standard one within some given error bars. However, monitoring the oscillation long enough we can determine the value of $N_{ph}$, provided $N_{ph} < \infty$.

For a finite $N$, the eigenvalues $s/N$ are distributed around the most probable value $s/N \approx Z_\omega$ (i.e. $s \approx N_{ph} \chi_\omega$ or simply $s \approx N_{ph}$ in optical region), resulting in non-trivial distribution of Rabi frequencies even in exact vacuum. In consequence, instead of a single-frequency oscillation we obtain beats analogous to those occurring in Rabi oscillations in the presence of a coherent light. Rabi frequency in exact resonance and in a subspace characterized by $s$ equals

$$2g\sqrt{s} = 2g\sqrt{\omega} \sqrt{\frac{s}{N\omega}} = 2g_{ph} \sqrt{\frac{s}{N_{ph}}},$$

so that the most probable Rabi frequency is $2g_{ph}$. The first revival of a collapsed vacuum Rabi oscillation will have its maximum when phases of neighboring and dominating oscillating terms differ by the factor of $2\pi$. This means that the revival time, $t_r$, can be found from the following relation

$$2g_{ph} t_r - 2g_{ph} \sqrt{\frac{N_{ph} - 1}{N_{ph}}} t_r = 2\pi$$

or

$$t_r = \left( N_{ph} + \sqrt{N_{ph}(N_{ph} - 1)} \right) \pi / g_{ph}.$$

Figure 5.3: Comparison of $\Omega(t)^{\text{err}}$ (dashed) and $\Omega(t)$ (solid), for (A) $N_{ph} = 10$ and (B) $N_{ph} = 400$ (curves are indistinguishable). The other parameters are: $N = 10^5$, $g_{ph} = 47\pi 10^3$ Hz, $\gamma_{1,ph} = \gamma_{2,ph} = 0.1g_{ph}$, $\gamma_3 = 0.001g_{ph}$.
If we take dissipation into account, the revival can be seen only if the oscillation occurring in (5.32) is still visible. The amplitude of oscillation is described, at \( s \approx N_{ph} \), by

\[
\varepsilon = \frac{1}{2} e^{-\frac{\gamma_{1,ph}+\gamma_{2,ph}+\gamma_{3}}{4} t_r},
\]

or

\[
\gamma = \gamma_{1,ph} + \gamma_{2,ph} + \gamma_{3} = -\frac{4}{t_r} \ln 2\varepsilon,
\]

with \( 0 < \varepsilon < 1/2 \). The dependence of \( \gamma \) on \( \varepsilon \) is shown in figure 5.4. Figure 5.5 shows the revival of the decayed \( p_g(t) \). The parameters used in figure 5.1 imply \( \varepsilon \sim 10^{-20} \), so the effect would not be visible in experiments where error bars grow with time similarly to those from [11].

### 5.5 Finite \( N \) or \( N \rightarrow \infty? \)

The application of weak law of large numbers, \( N \rightarrow \infty \), replaces fractions \( s/N \) by probabilities \( Z_\omega \). As has been discussed earlier, this leads the correspondence principle with standard regularized quantum optics, a fact that guarantees that the new theory can be regarded as a generalization of the standard one. The passage of one theory into the other should be meant literally, so that, in particular, the vacuum energy is left divergent. At the level of representation, the weak law follows from spectral representations of the frequency of success operator \( L_\omega \) occurring at right-hand sides of the commutator \([a_\omega, a_{\omega'}^\dagger] = \delta_{\omega\omega'}L_\omega \). The reducible representation may be regarded as a “quantized” form of the standard naively regularized irreducible representation \([a_\omega, a_{\omega'}^\dagger] = \delta_{\omega\omega'}Z_\omega 1 \), where \( Z_\omega \) is a regularizing function. The naive regularization is known to be in conflict with Poincaré covariance of canonical commutation relations. The replacement of the function \( Z_\omega \) by the operator \( L_\omega \) leads, on
Figure 5.5: Probability of the atomic ground state as a function of the effective time $t$ for $\gamma_1 = \gamma_2 = 83.912$, $\gamma_3 = 10$, and $N_{ph} = 400$ (these parameters yield $\varepsilon \approx 0.23$). The revival time (5.43), $t_r \approx 0.017s$, is indicated by the vertical line. It approximately determines the moment of maximal visibility of the revival.

On the other hand, to the correct behavior of the commutator under Poincaré transformations [1, 3, 70, 71].

Now, let us take a closer look at the thermodynamic limit $N \to \infty$ with $N_{ph} = NZ = \text{const}$ (notice that in terms of physical $N$, $N_{ph}$, a correspondence principle is obtained for $N_{ph} \to \infty$). The constancy of $N_{ph}$ implies $Z = N_{ph}/N \to 0$. Since $Z$ is defined as the maximum of the set of $Z_\omega$'s, we immediately get $Z_\omega \to 0$ for all $\omega$'s with normalization condition $\sum_\omega Z_\omega = 1$. All these facts leads to a conclusion that $\chi_\omega = Z_\omega/Z \to 1$, which practically shifts our automatic cutoff to infinity (but does not spoil normalizability of vacuum states).

Thermodynamic limits typically make some quantities divergent whereas some other quantities are well defined (a good example is a glass of water, where the density of water behaves correctly in the limit, but the water mass becomes infinite). A similar situation is encountered in the proposed treatment of cavity QED. The probability $p_g(t)$ becomes well defined even if $N$ tends to infinity with $N_{ph}$ kept constant, but the vacuum energy diverges. Indeed

$$\sum_\omega \frac{\hbar \omega}{2} NZ_\omega = N_{ph} \sum_\omega \frac{\hbar \omega}{2} \chi_\omega \to \infty,$$

as $\chi_\omega \to 1$ for arbitrary $\omega$. So, physically $N$ is required to be large but finite, while $Z$ should be small but nonzero, although vacuum Rabi oscillation is well defined even for $N_{ph} \to \infty$.

Furthermore, in the thermodynamic limit the bare coupling constant is

$$g = \frac{g_{ph}}{\sqrt{Z}} = g_{ph} \sqrt{\frac{N}{N_{ph}}},$$

(5.44)

The Jaynes-Cummings interaction
\[ gR_+a_\omega + \text{H.c.} = g_{\text{ph}} \sqrt{\frac{N}{N_{\text{ph}}}} R_+ \frac{1}{\sqrt{N}} (a_\omega \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes a_\omega) + \text{H.c.} \]
\[ = \frac{g_{\text{ph}}}{\sqrt{N_{\text{ph}}}} R_+ (a_\omega \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes a_\omega) + \text{H.c.,} \quad (5.44) \]

shows that the physical coupling parameter is, in fact, independent of \( N \) (still, the atom can interact with \( N \) field-oscillators). Let us stress, that all numerical experiments show quick convergence, with growing \( N \), of \( p_g(t) \) to a function whose shape is characterized by \( N_{\text{ph}} \).
Chapter 6

Some open questions

It would be interesting to analyze the other experiments involving finite-level atoms, especially those with masers and mazers [31, 34, 72, 73] but a technical difficulty is that exact solutions are not there available at the moment. The experiments testing spectra of light [74, 75, 76] in cavity QED are next realistic goals in this context. We have already computed the vacuum Rabi splitting [77, 78], with the conclusion that $N \to \infty$ reconstructs the standard results, which is yet another example of the correspondence principle.

The second, besides Rabi oscillation, phenomenon commonly considered as proving the quantumness of light is the three-peaked spectrum of resonance fluorescence [79, 80]. Although detailed results are not yet ready, it can be said, however, that the problem seems to be reducible to subproblems that may be dealt with the known, standard techniques. This is the case for the spontaneous emission problem that has been solved within the Weisskopf-Wigner approximation and led to renormalized expressions for the decay coefficients and the Lamb shift. All these partial results were not included in this thesis, and remain unpublished.

Another aspect, perhaps worthy of a more detailed study is the meaning of the parameter $\gamma_3$ responsible for long-wave transitions in open cavities. System-reservoir interactions of the form $\beta a^\dagger a \otimes B$, i.e. with the term $\alpha(a + a^\dagger) \otimes B$ absent, are characterized by $\gamma_1 = \gamma_2 = 0$, and a non-zero $\gamma_3$. The resulting master equation at zero temperature transforms, in irreducible representations of CCR, the initial product state $|e, 0\rangle$ into $1/\sqrt{2}(|g, 1\rangle - |e, 0\rangle)$, the maximally entangled EPR state. In reducible representations the resulting entangled state involves $N + 1$ qubits. Such processes are possible only in formalisms where Davies operators are defined in the dressed-state basis. The usual formalism results in damping towards bare, hence, separable states.

The simplest really nontrivial target is to discuss those very few exactly solvable renormalizable models of quantum field theory where all infrared and ultraviolet divergences do occur in the standard treatments. Of particular interest are the models of Lee [81] and Thirring [82, 83]. The work is in progress.
The derivation presented in this Appendix was suggested to us by R. Alicki. Let us start with the Hamiltonian

\[ H = \hbar \Omega_S + \hbar \Omega_R + \hbar \Omega_{\text{int}}, \]  

(6.1)

where \( \hbar \Omega_S \) and \( \hbar \Omega_R \) are Hamiltonians of the system and the reservoir, respectively. The system-reservoir interaction is assumed in the form

\[ \Omega_{\text{int}} = A \otimes B = \left\{ \alpha (a + a^\dagger) + \beta a^\dagger a \right\} \otimes \sum_k g_k (b_k + b_k^\dagger), \]  

(6.2)

with \( \alpha \) and \( \beta \) being parameters. The term proportional to \( \alpha \) is the standard one. The additional part, proportional to \( \beta \), is the new term proposed by Alicki. Following the procedure described in [38] we perform the decomposition of the operator \( A \):

\[ A = \sum_\omega A(\omega), \]  

(6.3)

\[ A(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi(\epsilon) A \Pi(\epsilon'). \]  

(6.4)

\( \Pi(\epsilon) \) is the projector on the eigensubspace corresponding to the eigenvalue \( E = \hbar \epsilon \) of \( H_S \). Then

\[ [\Omega_S, A(\omega)] = -\omega A(\omega), \]  

(6.5)

\[ [\Omega_S, A^\dagger(\omega)] = +\omega A^\dagger(\omega). \]  

(6.6)

and \( A^\dagger(\omega) = A(-\omega) \). Following the standard steps [38] we arrive at

\[ \dot{\rho}(t) = -i[H_S, \rho(t)] + \sum_{\omega > 0} \gamma(\omega) \left( A(\omega) \rho(t) A^\dagger(\omega) - \frac{1}{2} [A^\dagger(\omega) A(\omega), \rho(t)]_+ \right) \]

\[ + \sum_{\omega > 0} \gamma(-\omega) \left( A^\dagger(\omega) \rho(t) A(\omega) - \frac{1}{2} [A(\omega) A^\dagger(\omega), \rho(t)]_+ \right), \]  

(6.6)

where we have ignored the energy shifts caused by the environment and assumed \( \gamma(0) = 0 \) (which is true for the usual models of thermal reservoirs [38]).

Now assume that \( H_S \) is the Jaynes-Cummings Hamiltonian in exact resonance. The eigenstates of \( H_S \) are the dressed states

\[ |\Omega_{N, \pm}\rangle = \frac{1}{\sqrt{2}} (|g, N\rangle \pm |e, N - 1\rangle). \]  

(6.7)
The annihilation operators for the atom-field system are given explicitly by

\[
A(\Omega_{N',m'} - \Omega_{N,m}) = \Pi(\Omega_{N,m}) \left\{ \alpha (a + a^\dagger) + \beta a^\dagger a \right\} \Pi(\Omega_{N',m'})
\]

\[
= \frac{1}{2} \alpha \left( \sqrt{N+1} + mm' \sqrt{N} \right) \delta_{N',N+1|\Omega_{N,m}} \langle \Omega_{N+1,m'} \rangle
\]

\[
+ \frac{1}{2} \alpha \left( \sqrt{N} + mm' \sqrt{N-1} \right) \delta_{N',N-1|\Omega_{N,m}} \langle \Omega_{N-1,m'} \rangle
\]

\[
+ \frac{1}{2} \beta \left( N + mm' (N - 1) \right) \delta_{N',N|\Omega_{N,m}} \langle \Omega_{N,m'} \rangle.
\] (6.5)

To proceed further let us consider the two special cases, \( N = 0 \) and \( N = 1 \). For \( N = 0 \),

\[
A(\Omega_{1,\pm} - \Omega_0) = \frac{1}{\sqrt{2}} \alpha |\Omega_0 \rangle \langle \Omega_{1,\pm}|.
\] (6.6)

For \( N = 1 \) we get

\[
A(\Omega_{N',m'} - \Omega_{1,m}) = \frac{1}{2} \alpha \delta_{N',2} \left( \sqrt{2} + mm' 1 \right) |\Omega_{1,m} \rangle \langle \Omega_{2,m'}|
\]

\[
+ \frac{1}{\sqrt{2}} \alpha \delta_{N',0} |\Omega_{1,m} \rangle \langle \Omega_0|
\]

\[
+ \frac{1}{2} \beta \delta_{N',1} |\Omega_{1,m} \rangle \langle \Omega_{1,m'}|.
\] (6.5)

Keeping \( N = 1 \) and substituting explicit values for \( N', m \) and \( m' \) we get the following operators:

\[
A(\Omega_{2,+} - \Omega_{1,+}) = \frac{1}{2} \alpha (\sqrt{2} + 1) |\Omega_{1,+} \rangle \langle \Omega_{2,+}|,
\] (6.6)

\[
A(\Omega_{2,+} - \Omega_{1,-}) = \frac{1}{2} \alpha (\sqrt{2} - 1) |\Omega_{1,-} \rangle \langle \Omega_{2,+}|,
\] (6.7)

\[
A(\Omega_{2,-} - \Omega_{1,+}) = \frac{1}{2} \alpha (\sqrt{2} - 1) |\Omega_{1,+} \rangle \langle \Omega_{2,-}|,
\] (6.8)

\[
A(\Omega_{2,-} - \Omega_{1,-}) = \frac{1}{2} \alpha (\sqrt{2} + 1) |\Omega_{1,-} \rangle \langle \Omega_{2,-}|,
\] (6.9)

\[
A(\Omega_0 - \Omega_{1,\pm}) = \frac{1}{\sqrt{2}} \alpha |\Omega_{1,\pm} \rangle \langle \Omega_0|,
\] (6.10)

\[
A(\Omega_{1,\pm} - \Omega_{1,+}) = \frac{1}{2} \beta |\Omega_{1,+} \rangle \langle \Omega_{1,\pm}|,
\] (6.11)

\[
A(\Omega_{1,\pm} - \Omega_{1,-}) = \frac{1}{2} \beta |\Omega_{1,-} \rangle \langle \Omega_{1,\pm}|,
\] (6.12)

\[
A(\Omega_{1,-} - \Omega_{1,+}) = \frac{1}{2} \beta |\Omega_{1,+} \rangle \langle \Omega_{1,-}|,
\] (6.13)

\[
A(\Omega_{1,-} - \Omega_{1,-}) = \frac{1}{2} \beta |\Omega_{1,-} \rangle \langle \Omega_{1,-}|.
\] (6.14)

In the simplest case \( T = 0 \), and with the initial condition we have worked with above, we
\[ \dot{\rho}(t) = -i[\Omega_S, \rho(t)] \]
\[ + \gamma(\Omega_{1+} - \Omega_0)\alpha^2 \left( \frac{1}{2} |\Omega_0\rangle \langle \Omega_+| \langle \rho(t)| \Omega_+ \rangle \langle \Omega_0| - \frac{1}{4} |\Omega_+\rangle \langle \Omega_+|, \rho(t) \rangle \langle \Omega_+| + \right) \]
\[ + \gamma(\Omega_{1-} - \Omega_0)\alpha^2 \left( \frac{1}{2} |\Omega_0\rangle \langle \Omega_-| \langle \rho(t)| \Omega_- \rangle \langle \Omega_0| - \frac{1}{4} |\Omega_-\rangle \langle \Omega_-|, \rho(t) \rangle \langle \Omega_-| + \right) \]
\[ + \gamma(\Omega_{1+} - \Omega_{1-})\beta^2 2 \left( \frac{1}{2} |\Omega_-\rangle \langle \Omega_+| \langle \rho(t)| \Omega_+ \rangle \langle \Omega_-| - \frac{1}{4} |\Omega_+\rangle \langle \Omega_+|, \rho(t) \rangle \langle \Omega_+| + \right) ). \]

And this is the equation we have postulated, if \( \gamma_1 = \gamma(\Omega_{1+} - \Omega_0)\alpha^2 \), \( \gamma_2 = \gamma(\Omega_{1-} - \Omega_0)\alpha^2 \), and \( \gamma_3 = \gamma(\Omega_{1+} - \Omega_{1-})\beta^2 /2 \). The extension to \( T > 0 \) is obvious.
Bibliography


[57] Private communication.


